’Tis Whole

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1 Solution with consecutive numbers

The total sum of integers from 1 to n is equal to

$$1 + 2 + 3 + \cdots + n = \frac{n(1+n)}{2}. \tag{1}$$

Let:

1. x be the first of three consecutive numbers;
2. $x + 1$ be the second of three consecutive numbers; and
3. $x + 2$ be the third of three consecutive numbers.

$\therefore$

$$\frac{\frac{n(1+n)}{2} - x - (x+1) - (x+2)}{n-3} = 7.5,$$

$$\frac{\frac{n(1+n)}{2} - 3x - 3}{n-3} = 7.5,$$

$$\frac{\frac{n(1+n)}{2} - \frac{6x+6}{2}}{n-3} = 7.5,$$

$$\frac{\frac{n^2+n-6x-6}{2}}{n-3} = 7.5,$$

$$\frac{n^2 + n - 6x - 6}{2n-6} = 7.5,$$

$$n^2 + n - 6x - 6 = 15n - 45,$$

$$n^2 - 14n - 6x + 39 = 0,$$

$$\frac{n^2 - 14n + 39}{6} = x. \tag{2}$$

The smallest value of the first of the three consecutive numbers is 1. The largest value of the first of the three consecutive numbers is $n - 2$. Therefore,

$$1 \leq x \leq n - 2. \tag{3}$$

Combining equation (2) with inequality (3), then

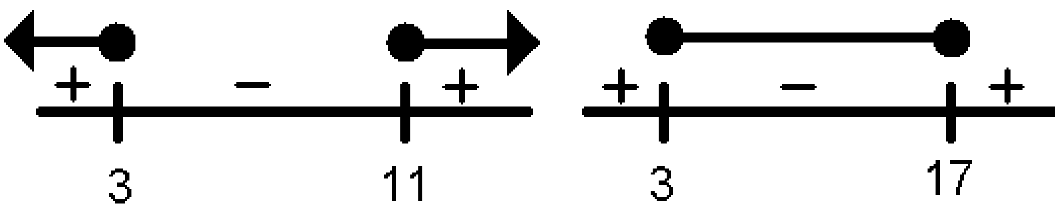
$$1 \leq \frac{n^2 - 14n + 39}{6} \leq n - 2,$$

$$1 \leq \tfrac{n^2-14n+39}{6} \quad \wedge \quad \tfrac{n^2-14n+39}{6} \leq n - 2,$$

$$6 \leq n^2 - 14n + 39 \quad \wedge \quad n^2 - 14n + 39 \leq 6n - 12,$$

$$0 \leq n^2 - 14n + 33 \quad \wedge \quad n^2 - 20n + 51 \leq 0,$$

$$0 \leq (n - 11)(n - 3) \quad \wedge \quad (n - 3)(n - 17) \leq 0.$$



$\therefore$

$$n = 3 \vee 11 \leq n \leq 17.$$

However, $n = 3$ is not a valid solution, because after removing three consecutive numbers there would not be any numbers left to yield an average of 7.5. Therefore,

$$11 \leq n \leq 17. \tag{4}$$

Using again equation (2), next table contains values of x for all integer values of n between 11 and 17.

n	11	12	13	14	15	16	17
x	1	2.5	4.3	6.5	9	11.83	15

Since x has to be a whole number, then the only n values that are valid are: 11 (with 1, 2, and 3 as the removed numbers), 15 (with 9, 10, and 11 as the removed numbers), and 17 (with 15, 16, and 17 as the removed numbers).

2 Solution with non-consecutive numbers

Let a, b and c be the three removed numbers.

Using again equation (1), then

$$\frac{\frac{n(1+n)}{2} - a - b - c}{n-3} = 7.5,$$

$$\frac{\frac{n(1+n)}{2} - \frac{2a}{2} - \frac{2b}{2} - \frac{2c}{2}}{n-3} = 7.5,$$

$$\frac{\frac{n^2+n-2a-2b-2c}{2}}{n-3} = 7.5,$$

$$\frac{n^2+n-2a-2b-2c}{2n-6} = 7.5,$$

$$n^2 + n - 2a - 2b - 2c = 15n - 45,$$

$$n^2 + n - 15n + 45 = 2a + 2b + 2c,$$

$$n^2 - 14n + 45 = 2\left(a + b + c\right),$$

$$\frac{n^2 - 14n + 45}{2} = a + b + c. \qquad (5)$$

The smallest value of the sum of the three numbers is 6, when the removed numbers are 1, 2, and 3. The largest value of the sum of the three numbers is $n + (n-1) + (n-2)$. Therefore,

$$6 \leq a + b + c \leq n + (n-1) + (n-2),$$

$$6 \leq a + b + c \leq 3n - 3. \qquad (6)$$

Combining equation (5) with inequality (6), then

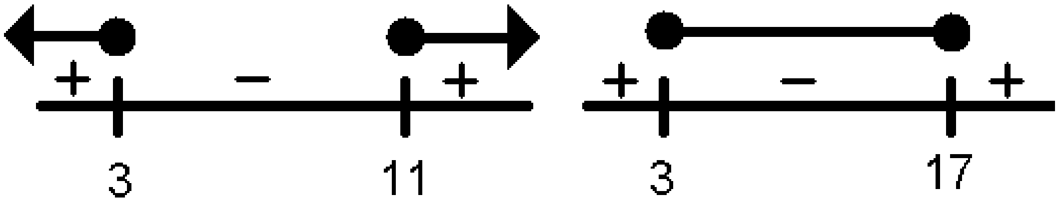
$$6 \leq \frac{n^2 - 14n + 45}{2} \leq 3n - 3,$$

$$6 \leq \frac{n^2-14n+45}{2} \quad \wedge \quad \frac{n^2-14n+45}{2} \leq 3n - 3,$$

$$12 \leq n^2 - 14n + 45 \quad \wedge \quad n^2 - 14n + 45 \leq 6n - 6,$$

$$0 \leq n^2 - 14n + 33 \quad \wedge \quad n^2 - 20n + 51 \leq 0,$$

$$0 \leq (n-11)(n-3) \quad \wedge \quad (n-3)(n-17) \leq 0.$$



$\therefore$

$$n = 3 \vee 11 \leq n \leq 17.$$

However, $n = 3$ is not a valid solution, because after removing three consecutive numbers there would not be any numbers left to yield an average of 7.5. Therefore,

$$11 \leq n \leq 17. \tag{7}$$

Using again equation (5), next table contains values of $a + b + c$ for all integer values of n between 11 and 17.

n	11	12	13	14	15	16	17
$a + b + c$	6	10.5	16	22.5	30	38.5	48

Since $a + b + c$ has to be a whole number, because a, b, c also have to be whole numbers, then the only n values that are valid are: 11, 13, 15, and 17.