Nrich: Conjugate Tracker

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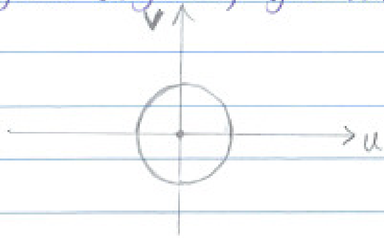
In the equation $x^2+px+q=0$, q is fixed while p is varied. As p changes, the complex roots of the equation also change. If these complex roots are plotted on to an Argand diagram, you will see that the roots lie on a circle.



This is what the diagram should look like, where a root z is split into its real and complex components, $u+vi$.

To prove that the complex roots do lie on a circle, let the two roots of the equation be $z_1 = u+vi$ and $z_2 = u-vi$, since they are always reflections of each other in the real axis (u-axis).

It is known that $z_1 \times z_2 = q$. This is because if the quadratic formula is used to solve the equation $x^2+px+q=0$, z_1 and z_2 can be expressed as follows:

$$z_1 = \frac{-p+\sqrt{p^2-4q}}{2} = -\frac{p}{2} + \frac{\sqrt{4q-p^2}}{2}i$$

$$z_2 = \frac{-p-\sqrt{p^2-4q}}{2} = -\frac{p}{2} - \frac{\sqrt{4q-p^2}}{2}i$$

$$z_1 \times z_2 = \left(-\frac{p}{2} + \frac{\sqrt{4q-p^2}}{2}i\right)\left(-\frac{p}{2} - \frac{\sqrt{4q-p^2}}{2}i\right) = \left(-\frac{p}{2}\right)^2 - \left(\frac{\sqrt{4q-p^2}}{2}i\right)^2$$

$$= \frac{p^2}{4} + \frac{4q-p^2}{4} = \frac{4q}{4} = q$$

Since $z_1 \times z_2 = q$, we can substitute the original values of z_1 and z_2 into here:

$$(u+vi)(u-vi) = q$$
$$u^2 - v^2i^2 = q$$
$$\therefore u^2+v^2 = q$$

So if the complex roots are plotted onto an Argand diagram with u- and v-axes, the curve formed will be a circle with centre $(0,0)$ and radius $\sqrt{q}$.