1. An ellipse with semi axes *a* and *b* fits between two circles of radii *a* and *b* (where b > a) as shown in the diagram. If the area of the ellipse is equal to the area of the annulus what is the ratio *b*:*a*?

The area of a circle is πr^2

The difference in areas of two circles of radii *b* and *a* is $\pi(b-a)^2$

The area of an ellipse is $\pi a b$

The question states that the area of the annulus is equal to that of the ellipse.

So, $\pi(b-a)^2 = \pi a b$

$(b-a)^2 = ab$

$b^2-a^2=ab$

Let $x = b{:}a = \frac{b}{a}$ $\quad xa = \frac{b}{a}\ a = b$

$(xa)^2 - a^2 = a\ (xa)$

$x^2a^2 - a^2 = a^2x$

$x^2a^2 - a^2 - a^2x = 0$

$a^2\ (x^2 - 1 - x) = 0$

$a^2\ (x^2 - x - 1) = 0$

All we're interested in is the value of x: that is, the ratio of b to a.

So, let's use the quadratic equation and solve for x:

$$x = \frac{-b \pm\sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-1)}}{2(1)}$$

$$= \frac{1\pm\sqrt{1+4}}{2}$$

$$= \frac{1\pm\sqrt{5}}{2}$$

The only positive solution for x is $\frac{1+\sqrt{5}}{2} = \varphi$

Thus the ratio of b:a must be φ.

2. Find the value of R if this sequence of 'nested square roots' continues indefinitely.

$$R = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}$$

Square roots are nasty things, so let's try squaring both sides:

$$R^2 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}^2$$

$$R^2 = 1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}$$

$$R^2 - 1 = \sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}$$

Does the right-hand side look familiar?

$$R^2 - 1 = R$$

$$R^2 - R - 1 = 0$$

We have this familiar quadratic equation again. Let's solve:

$$R = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1+4}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

The only positive solution for R is $\frac{1+\sqrt{5}}{2} = \varphi$