Stars

Can you find a relationship between the number of dots on the circle and the number of steps that will ensure that all points are hit?

The number of dots on the circle and the number of steps have to be relatively prime to ensure that every dot is hit. If these two numbers have a common factor greater than 1, the dots could be divided into groups, and each group could make a separate star. The shape of the star of each group is identical to that of a simplified group of numbers. For example, the five stars created when moving around a 40-point circle in steps of 15 are of identical shapes as the star created when moving around an 8-point ring in a step of 3. (As illustrated in Figure 1)

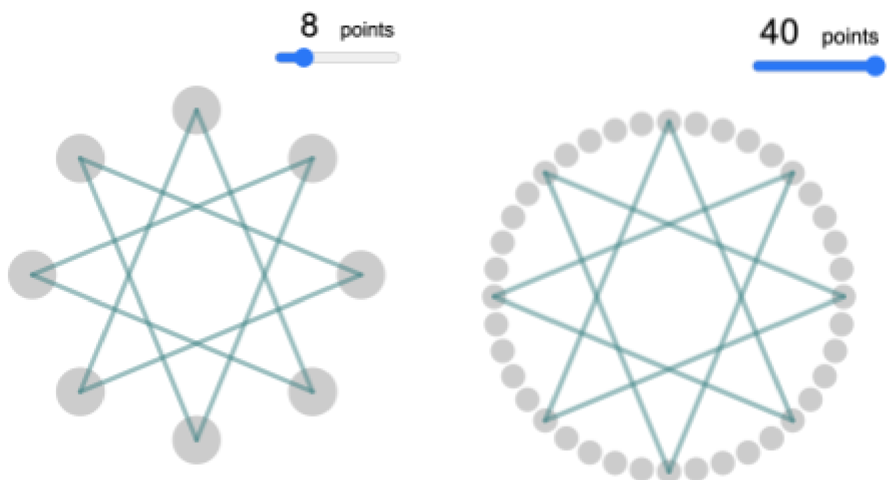

Figure 1

If the number of dots (n) and the number of steps (m) are relatively prime to each other, I will prove that the star will hit every point of the circle. Assume that we start drawing the star from a certain dot (Point Zero), and the drawing path returns to Point Zero after hitting a number of dots without any other dot being reached twice*. If n and m are relatively prime, the drawing path should hit all the dots.

We start proving this by looking at a simple example where n=5, m=2. Starting from point zero, we write down the distance that this path has covered, as is shown in Figure 2. The distance can be mapped to the dot number (0 - 4) by the Modulo operation with the divisor of 5. It is obvious that if Point Zero (dot of number 0) is reached again, the covered distance has to be a multiple of 5.

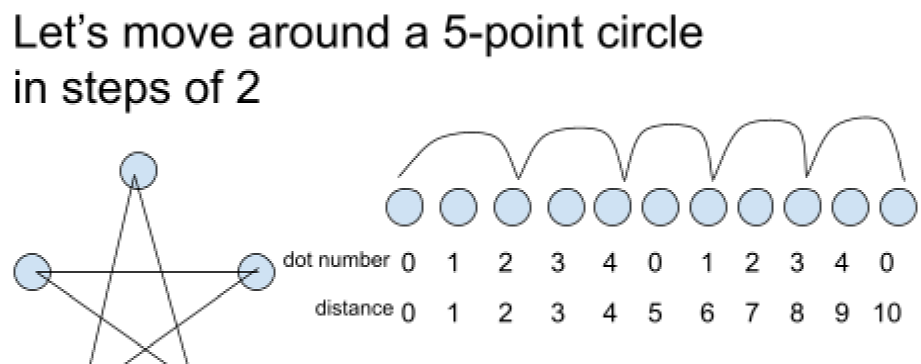

Fig. 2

On the other hand, because the number of steps is 2, the covered distance also has to be a multiple of 2. Therefore, the covered distance is a common multiple of 5 and 2. Since 5 and 2 are relatively prime, the covered distance is no less than 5*2. Besides, there is no other dot reached twice on this path, so the length of the path cannot be larger than 5*2. In conclusion, the length of the path has to be 5*2, the least common multiple (LCM) of 5 and 2, so all 5 dots have been reached with one go.

In a more general case of m dots and n steps, we can similarly find that if a path starts and ends at point 0, the length of the path has to be the LCM of both m and n, which is m*n if m and n are relatively prime.

However, if m and n are not relatively prime, the proof above doesn't hold because m*n is not their LCM. That means, the number of dots could be divided into k groups where k=HCF(m, n), and each group makes a star that is identical to the star created with m/k dots and n/k steps.

How can you work out what step sizes will visit all the points for any given number of points?

The step size that will visit all the points regardless of the number of them is ONE, because the HCF is always one between one and any other integer. This way the star can't be simplified, just like a fraction of one over anything can't be simplified.

Now consider 5 points. You can visit all the points irrespective of the step size. Which other numbers have this property?
Other numbers with this property are prime numbers because a prime number is relatively prime to all numbers smaller than it.

*If a certain dot was reached more than once in this path, we can define that dot as point 0