Last Biscuit

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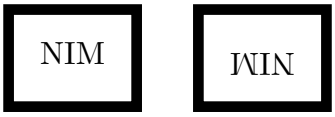
Before playing this, I knew the game of NIM and had played it many times with my friends and family. It began from 100 and both players were allowed to subtract any number from 1 to 9. Whoever would reach till 0 would win the game. This strategy involved thinking backwards. If the opponent is left anywhere from 1 to 9, then they can win. But at 10, they are certain to loose. So, I had to force the total to be at 10 when it's their turn. Now, my target to reach shifted from going to 10 instead of 0 at the moment. Again, If I want to force the opponent to leave them at 10, I can make that happen if I make them reach till 20. Continuing on, I would have to leave them at 90. Moving on last step, the total should be 100 and It should be the opponents turn for me to win.

For example, if they begin by saying 96 after subtracting 4, I can make the total go to 90 by subtracting 6. From 90, they subtract 7 and I subtract 3. It's 80 and their turn now. Chipping away multiples of 10 left the opponent at 10. Now, no matter what number from 1 to 9 they choose, I will definitely have to choose the supplementary of 10 to subtract from. If they subtract 3, I must subtract 7 on the next move to win!

Fun fact, if you rotate the word NIM by 180°, you get WIN



Moving from this version, to the last biscuit was a bit difficult. After playing this several time with the computer and my family, I was always stuck to a combination of 2 biscuits in one jar and 1 in the other, where it was a losing move for me. No matter what I do, I am certain to lose.

If I take out 1 from the jar with 2, the opponent would take 1 from each.
If I take out 2 from the jar with 2, the opponent would take 1 from the other jar

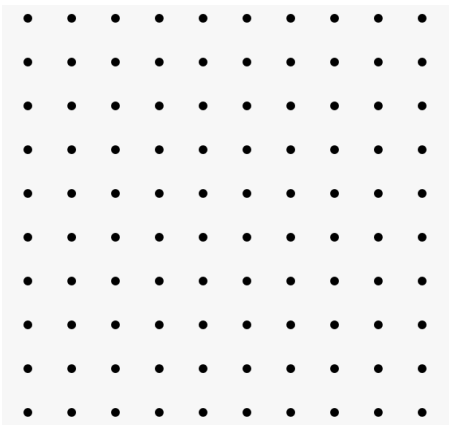
If I take out 1 from the jar with 1, then the opponent would take 2 from the other one. And if I take 1 from each, I still loose.

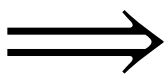
It was like the losing position of 10 in the game of NIM I knew about. Just like 10, 20, 30 and all multiples were a losing position, I tried other possible combinations where I was certainly going to loose. But, this was difficult, it was challenging to think of such combinations randomly. For the game of NIM, I visualised piles of 10 where I had to clear the entire pile after the opponent took any. And since I got to the strategy by thinking backwards, I thought of something similar for this too.

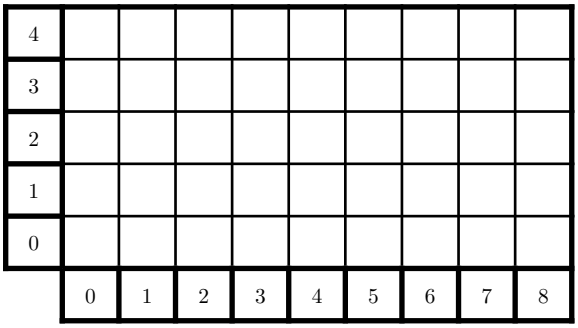
For convenience, let the biscuits left be represented as an ordered pair such as (4, 8) for the initial move.

Thinking backwards, if one jar was completely empty, then I can win by taking all the biscuits from the other one. Thus, winning moves were (a, 0) and (0, b). Also, if both piles had the same biscuits, then it was a winning move too! So (a, a) is also a winning move.

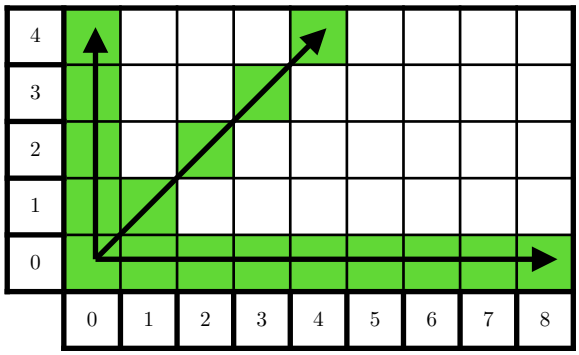
Thinking of ordered pairs and co-ordinates, I thought of visualising this problem as a graph! Instead of co-ordinates, it was better to use boxes and a grid which represented it much better.

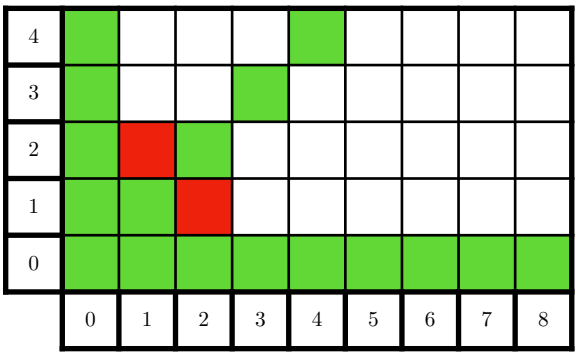




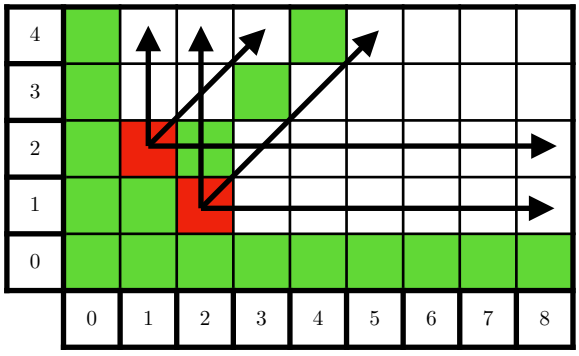


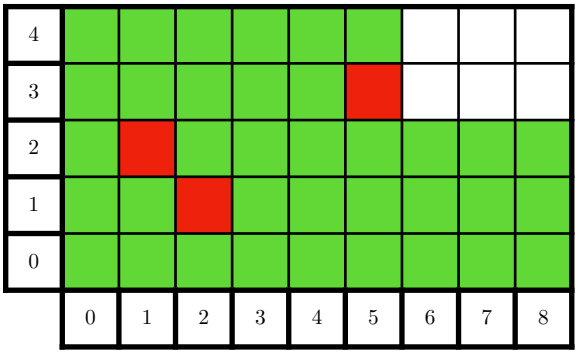
Now, I could represent my observations in this grid that I made. Green are the winning moves / positions here. Now, my observations that (1,2) and (2,1) are losing moves would be helpful here. I later found out that (1,3) and (3,1) are actually winning moves since now, I can force the opponent to (1,2) or (2,1) respectively.



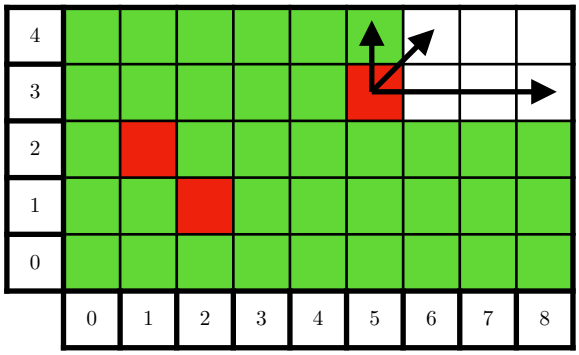


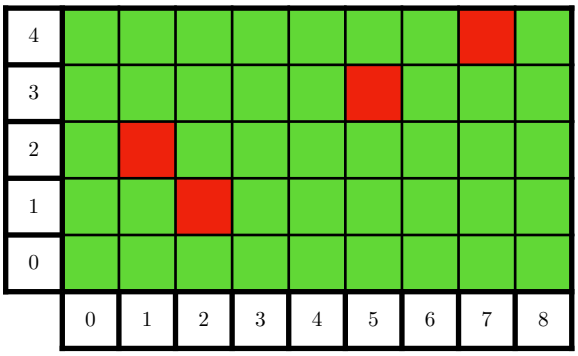
The pattern I see is to mark all the spots that are vertical, horizontal and diagonal to it as I have marked with the arrow. Similar to the simples version of NIM, now my targets are to make the opponent reach till the red spot. Hence, I can again think backward and mark vertical, horizontal and diagonal lines passing through it.





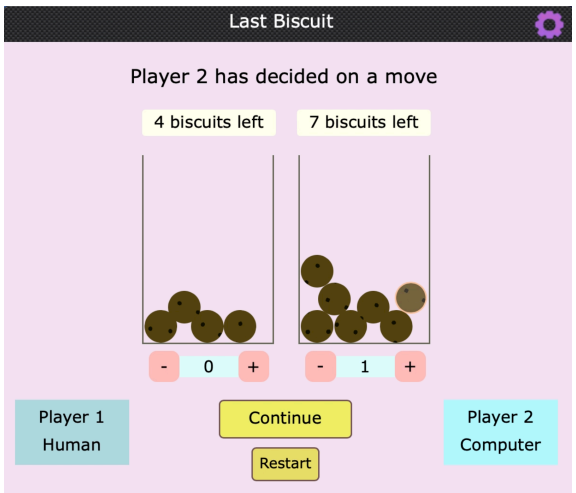
Now after spotting the pattern, I can deduce which spots are losing positions and which are for winning. Continuing this process, I've now marked all losing and winning spots.

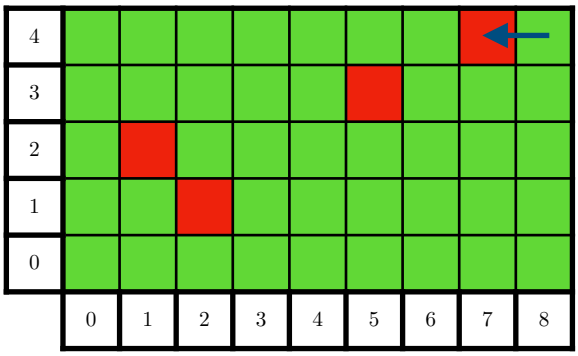


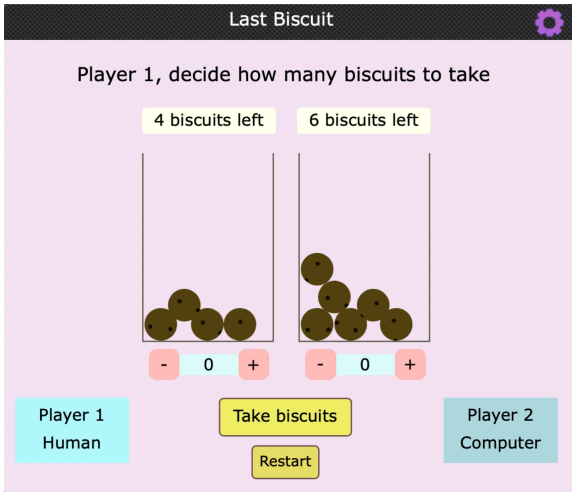


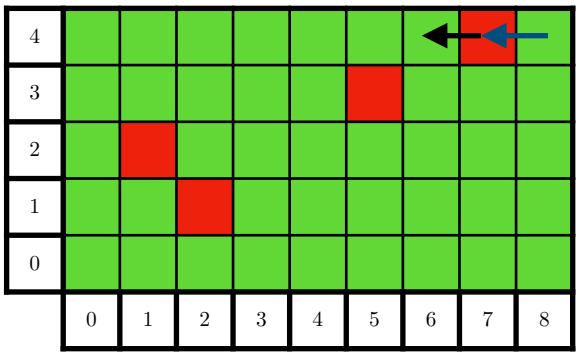
To show the game in action and how the strategy reveals, here are the moves I made with the computer alongside the visual representation of the game:

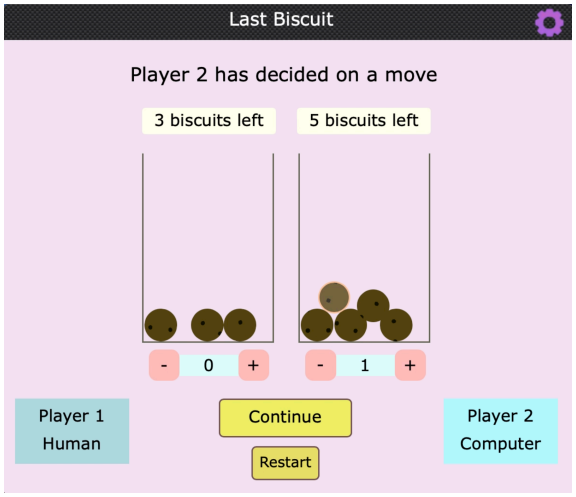
I began by forcing the computer into a losing position by taking away 1 biscuit from the jar with eight. Hence the game is now at (7, 4) with the computer's turn.

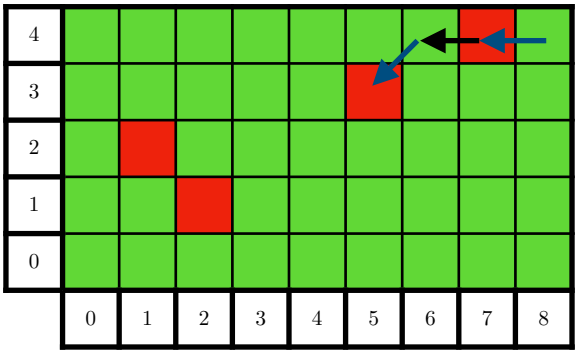


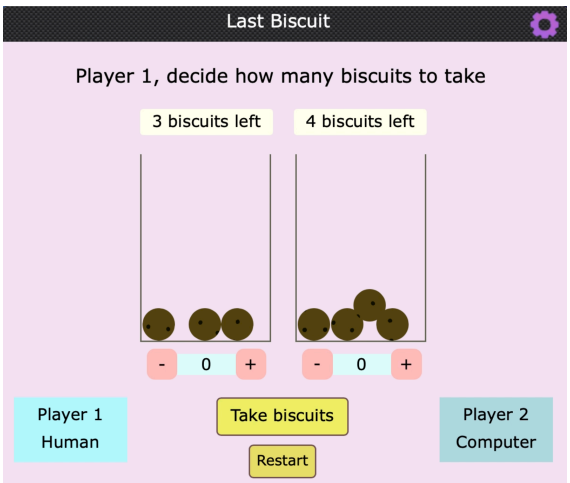


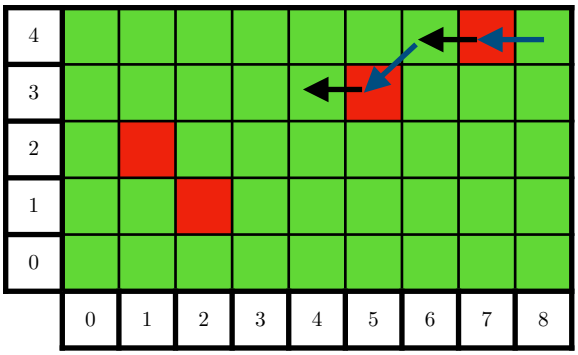


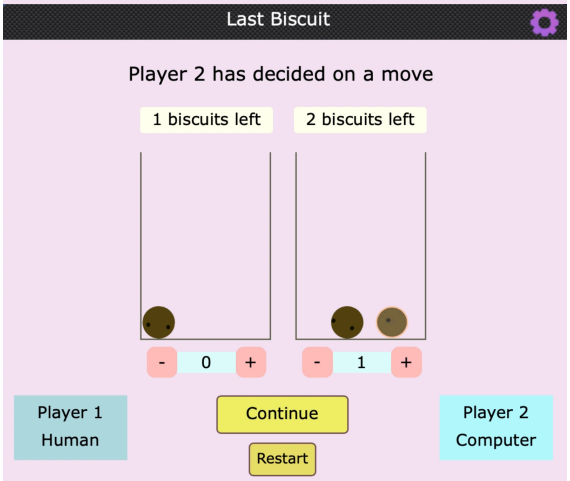


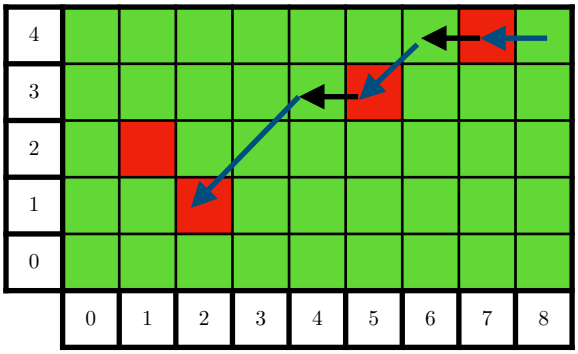


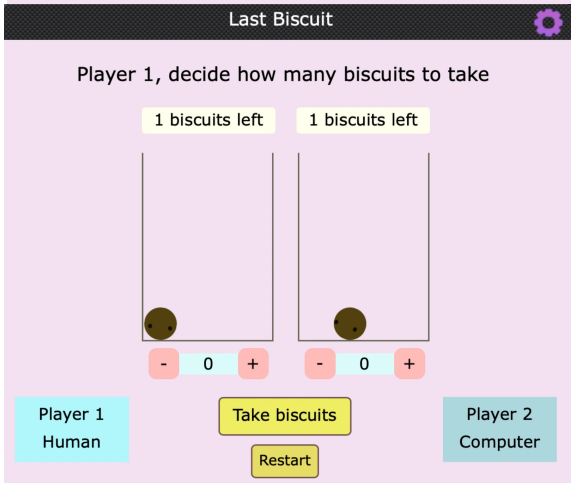


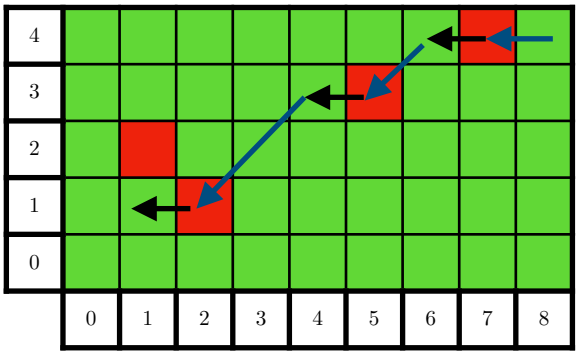

Last Biscuit
Player 1, decide how many biscuits to take
3 biscuits left
4 biscuits left
- 0 +
- 0 +
Player 1
Human
Take biscuits
Restart
Player 2
Computer

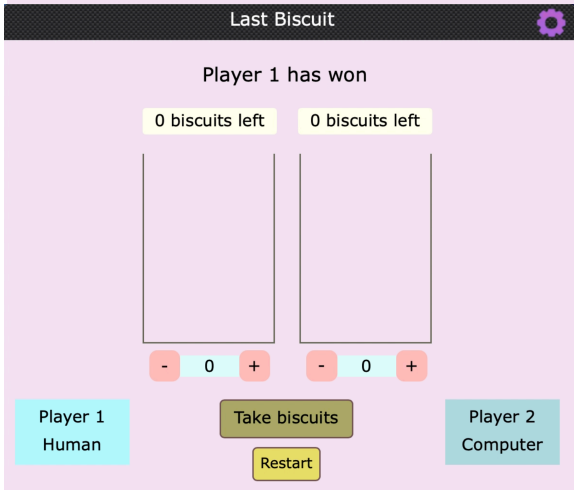

4
3
2
1
0
0 1 2 3 4 5 6 7 8

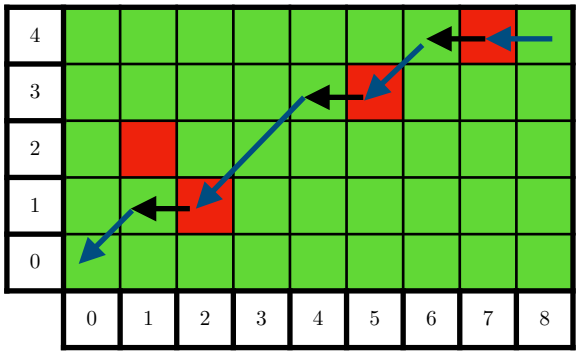

Last Biscuit
Player 2 has decided on a move
1 biscuits left
2 biscuits left
- 0 +
- 1 +
Player 1
Human
Continue
Restart
Player 2
Computer


4
3
2
1
0
0 1 2 3 4 5 6 7 8


Last Biscuit
Player 1, decide how many biscuits to take
1 biscuits left
1 biscuits left
- 0 +
- 0 +
Player 1
Human
Take biscuits
Restart
Player 2
Computer

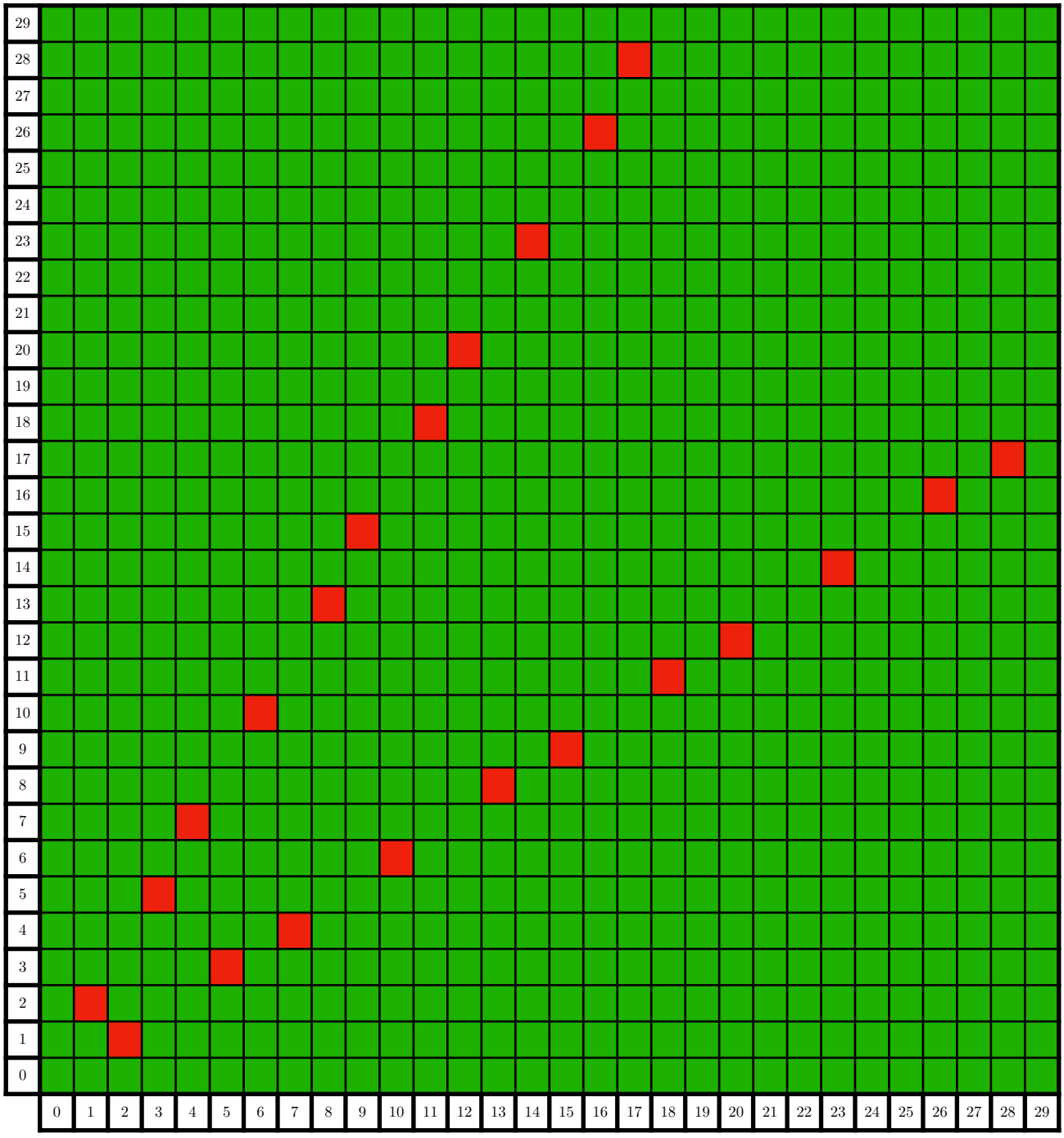

4
3
2
1
0
0 1 2 3 4 5 6 7 8


Last Biscuit
Player 1 has won
0 biscuits left
0 biscuits left
- 0 +
- 0 +
Player 1
Human
Take biscuits
Restart
Player 2
Computer

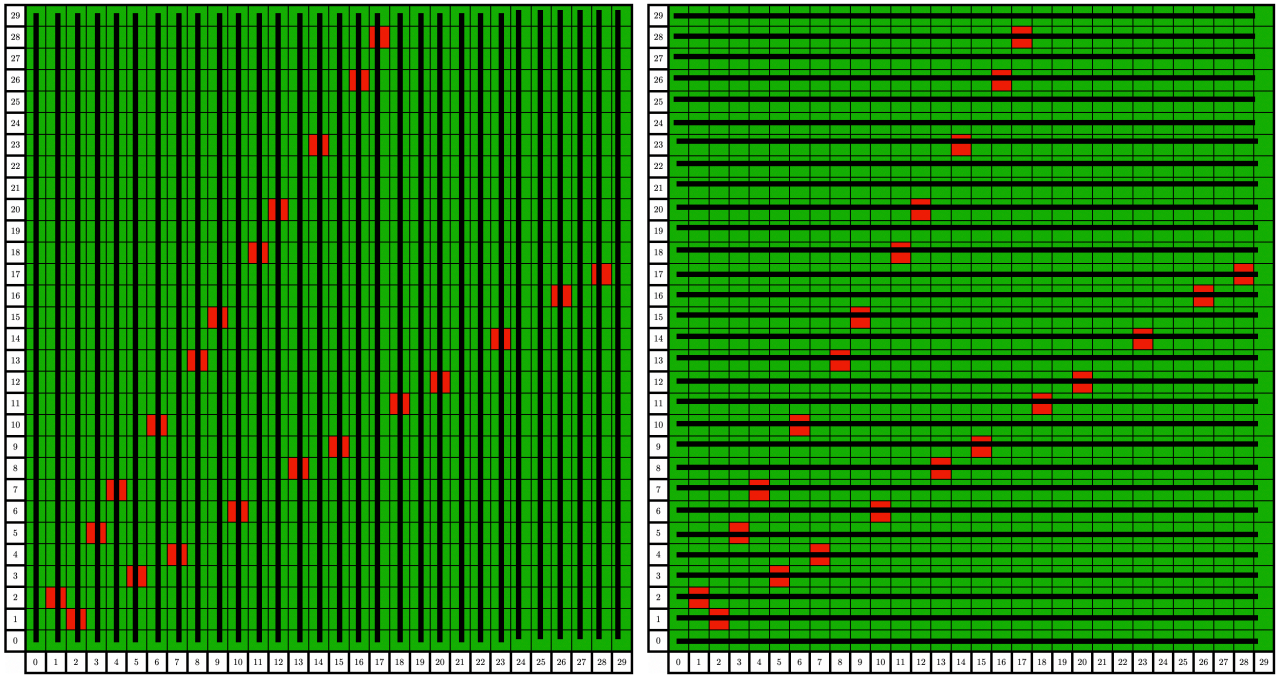

4
3
2
1
0
0 1 2 3 4 5 6 7 8

From this, I can conclude that the first player to begin with this arrangement of (4, 8) can always win. If however, the cookies were placed such that it is (4, 7), then the first player can always lose if the opponent plays strategically.

The obvious next step for me was to extend this grid to as large values as possible. Here is the result I got. This can be extended for finding the strategy with (100,50) as well.



I come across beautiful patterns and interesting observations. Such as, the formation of the squares along some particular slope that make up two lines, the symmetry along the line $y = x$ and maybe some pattern in the boxes. Maybe it forms a series as well? And there is only a single box in every horizontal and vertical line



From this, I also wondered about the variation of slightly different versions of this game. One could be the addition of an extra rule such as taking out multiples of 2 from one jar and 3 from the other. Or take ka from one jar and kb from the other one.

Interestingly, what would happen if we had 3 jars? Would the visualisation be 3 dimensional in a cube? And what about n jars then? That would be an elegant generalisation over the last biscuit!!