Always perfect

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Part 1a

I notice that the result is the square of the average of the two numbers that I took.

At first thought, I did the same as Claire. Taking $n(n+2)+1=(n+1)^2$

Interestingly, if we let the average to be n, then the two numbers which differ by 2, will be $n-1,\quad n+1$.

Multiplying the two and adding 1 gives this:

$(n-1)(n+1)+1=n^2-1+1=n^2$

A nice application of difference of two squares.

Part 1b

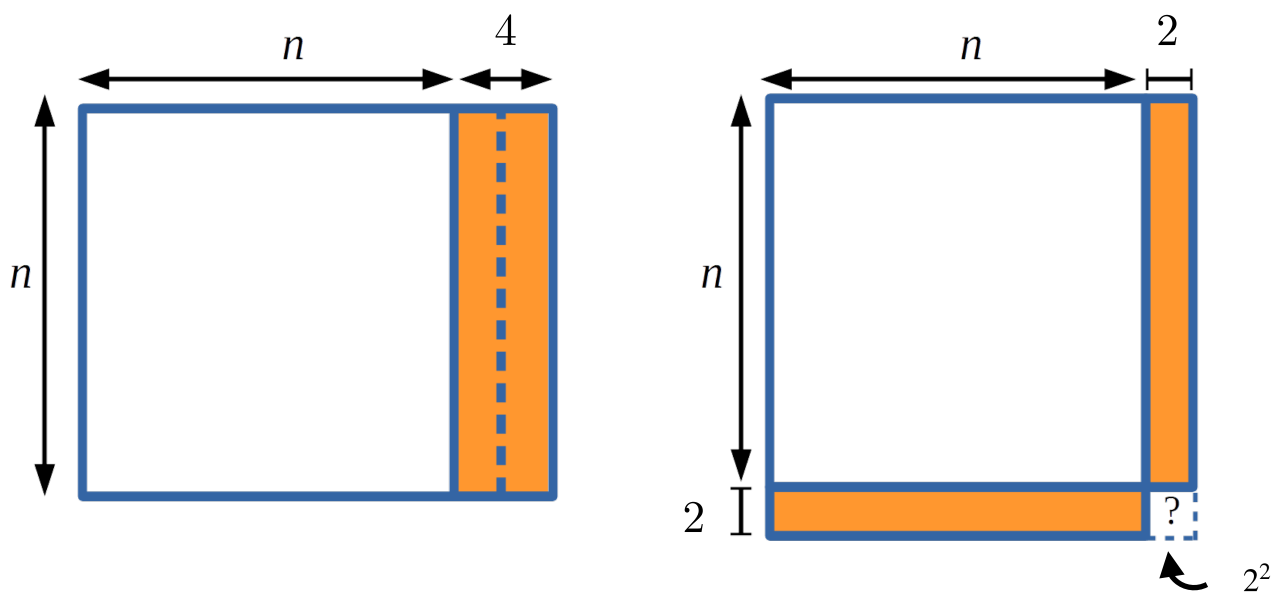
Armed with the difference of squares observation, I took the two numbers to be $n-2,\quad n+2$

Multiplying the two and adding 4 (because 2^2) gives this:

$(n-2)(n+2)+4=n^2-4+4=n^2$

Here n is the average of the two numbers that I took. For example, if the numbers which differ by four are 5 and 9, then the addition of their product and 4 would be 7 squared.

Similar to Charlie's method:



Similar to Claire's method:

$$(n)(n+4)+4 = n^2+4n+4$$
$$= (n+2)^2$$

Part 1c

Very similar, I get the pattern, YES!!

Part 1d

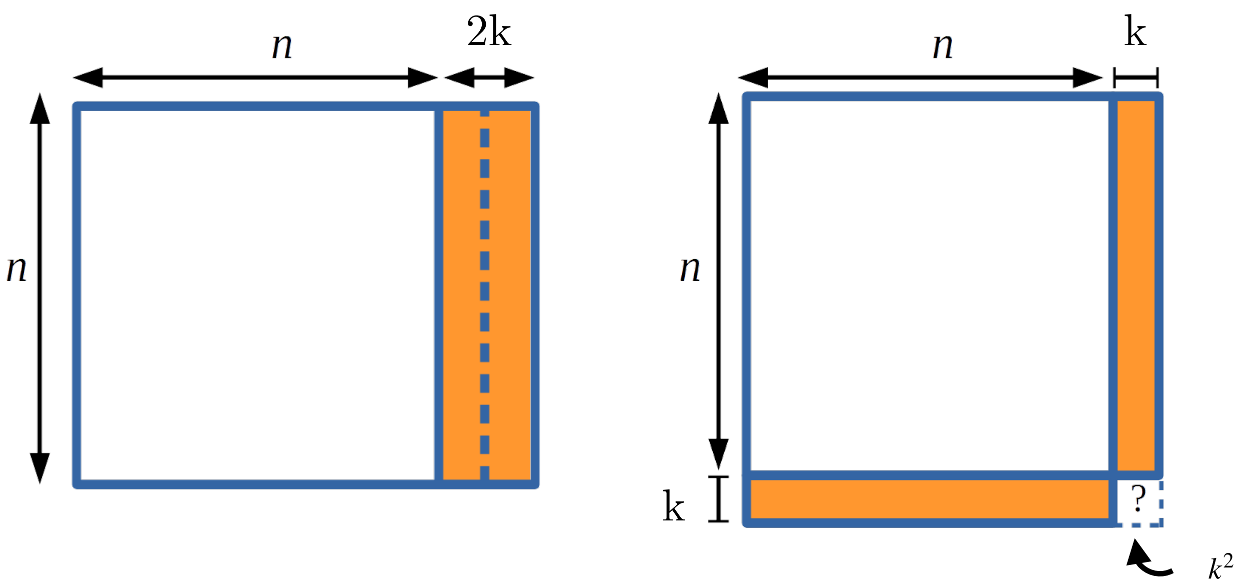
(1) My method:

Since the two numbers differ by $2k$, the numbers can be $n-k, \quad n+k$

Multiply the two numbers and add k^2

$$(n-k)(n+k)+k^2 = n^2-k^2+k^2$$
$$= (n)^2$$

(2) By Charlie's method:



(3) By Claire's method:

$$(n)(n+2k)+k^2 = n^2+2(n)(k)+k^2$$
$$= (n+k)^2$$

NOTE: The n in my method is different to the n in Claire's approach

Part 2(a)

$4 \times 5 \times 6 \times 7 + 1 = (28+1)^2$

$5 \times 6 \times 7 \times 8 + 1 = (40+1)^2$

$(n)(n+1)(n+2)(n+3)+1 = n^4 + 6n^3 + 11n^2 + 6n + 1$

	n^2	$3n$	1
n^2	n^4	$3n^3$	n^2
$3n$	$3n^3$	$9n^2$	$3n$
1	n^2	$3n$	1

$= (n^2 + 3n + 1)^2$

$(n+1)(n+2)(n+3)(n+4)+1 \Rightarrow \left((n+1)^2 + 3(n+1) + 1\right)^2$

$\left((4)^2 + 3(4) + 1\right)^2 = (28+1)^2$

Part 2(b)

$$2 \times 4 \times 6 \times 8 + 16 = 20^2$$

$$(n)(n+2)(n+4)(n+6) + 16 = x^4 + 12x^3 + 44x^2 + 48x + 16$$

	x^2	$6x$	4
x^2	x^4	$6x^3$	$4x^2$
$6x$	$6x^3$	$36x^2$	$24x$
4	$4x^2$	$24x$	16

$$= (x^2 + 6x + 4)^2$$

2(d)

Let the numbers differ by k

$\therefore$ numbers are: $n, n+k, n+2k, n+3k$

According to pattern observed, adding k^4 to the product will give a perfect square

$\therefore \quad (n)(n+k)(n+2k)(n+3k) + k^4$

$= n^4 + 6kn^3 + 11k^2n^2 + 6k^3n + k^4$

Factorize this quartic

	n^2	$3nk$	k^2
n^2	n^4	$3kn^3$	n^2k^2
$3nk$	$3kn^3$	$9n^2k^2$	$3nk^3$
k^2	n^2k^2	$3nk^3$	k^4

$= (n^2 + 3nk + k^2)^2$

$\therefore$ Product of four numbers differing by k added to k^4 is the square of $(n^2 + 3nk + k^2)$