Solutions to 'Always Perfect'

Part 1 (a)

Take n and $(n+2)$ as two numbers that differ by 2,

$$n(n+2)+1 = n^2+2n+1$$
$$= (n+1)^2$$

Part 1 (b)

Take n and $n+4$ as two numbers that differ by 4,

$$n(n+4)+4 = n^2+4n+4$$
$$= (n+2)^2$$

Part 1 (c)

Take n and $n+6$ as two numbers that differ by 6,

$$n(n+6)+9 = n^2+6n+9$$
$$= (n+3)^2$$

Part 1 (d)

For numbers differing by:
$$\left.\begin{array}{l} 2\text{, you add } 1 \\ 4\text{, you add } 4 \\ 6\text{, you add } 9 \end{array}\right\} \therefore \text{ when differs by } 2k\text{, add } k^2$$

$$n(n+2k)+k^2 = n^2+2kn+k^2$$
$$= (n+k)^2$$

(So you have to add k^2, where $2k$ is the difference between the two numbers.)

Part 2(a)

Write out a few examples:

$1 \times 2 \times 3 \times 4 + 1 = 25$

$2 \times 3 \times 4 \times 5 + 1 = 121$

$3 \times 4 \times 5 \times 6 + 1 = 361$

Notice, $(2 \times 3) - 1 = 5$, which when squared $= 25$

$(3 \times 4) - 1 = 11$, which when squared $= 121$

Stating this algebraically,

$$\left((n+1)(n+2) - 1\right)^2 = n(n+1)(n+2)(n+3)+1$$

We can prove this,

LHS: $(n^2 + 3n + 2 - 1)^2$

$= (n^2+3n+1)(n^2+3n+1)$

$= n^4 + 6n^3 + 11n^2 + 6n + 1$

RHS: $n(n+1)(n+2)(n+3)+1$

$= (n^2+n)(n+2)(n+3)+1$

$= (n^3+3n^2+2n)(n+3)+1$

$= n^4 + 3n^3 + 2n^2 + 3n^3 + 9n^2 + 6n + 1$

$= n^4 + 6n^3 + 11n^2 + 6n + 1$

LHS = RHS so must be true for all values of n.

Part 2(b)

Write out a few examples:

$$2 \times 4 \times 6 \times 8 + 16 = 400$$
$$3 \times 5 \times 7 \times 9 + 16 = 961$$
$$4 \times 6 \times 8 \times 10 + 16 = 1936$$

Notice, $(4 \times 6) - \sqrt{16} = 20$ which when squared $= 400$
$(5 \times 7) - \sqrt{16} = 31$ which when squared $= 961$

Stating this algerbraically:

$$((n+2)(n+4) - 4)^2 = n(n+2)(n+4)(n+6) + 16$$

We can prove this,

LHS: $(n^2 + 6n + 8 - 4)^2$
$= (n^2 + 6n + 4)(n^2 + 6n + 4)$
$= n^4 + 12n^3 + 44n^2 + 48n + 16$

RHS: $n(n+2)(n+4)(n+6) + 16$
$= (n^2 + 2n)(n+4)(n+6) + 16$
$= (n^3 + 6n^2 + 8n)(n+6) + 16$
$= n^4 + 6n^3 + 6n^3 + 36n^2 + 8n^2 + 48n + 16$
$= n^4 + 12n^3 + 44n^2 + 48n + 16$

LHS = RHS so must be true for all values of n.

Part 2(c)

Write out a few examples:

$$2 \times 5 \times 8 \times 11 + 81 = 961$$
$$3 \times 6 \times 9 \times 12 + 81 = 2025$$

Notice, $(5 \times 8) - 9 = 31$ which when squared $= 961$
$(6 \times 9) - 9 = 45$ which when squared $= 2025$

Stating this algerbraically:

$$((n+3)(n+6)-9)^2 = n(n+3)(n+6)(n+9) + 81$$

We can prove this,

LHS: $(n^2+9n+18-9)^2$
$= (n^2+9n+9)(n^2+9n+9)$
$= n^4+18n^3+99n^2+162n+81$

RHS: $n(n+3)(n+6)(n+9)+81$
$=(n^2+3n)(n+6)(n+9)+81$
$=(n^3+9n^2+18n)(n+9)+81$
$= n^4+9n^3+9n^3+18n^2+81n^2+162n+81$
$= n^4+18n^3+99n^2+162n+81$

LHS = RHS so must be true for all values of n.

Part 2(d)

For four numbers differing: by 1, you add 1
by 2, you add 16
by 3, you add 81

Notice that $1^4 = 1$
$2^4 = 16$
$3^4 = 81$

Where k = the difference between each of the 4 numbers, the number you add to the product of the four numbers is k^4

This can be stated as,

$$\left((n+k)(n+2k) - \sqrt{k^4}\right)^2 = n(n+k)(n+2k)(n+3k) + k^4$$

This can be proved,

LHS : $(n^2 + 3kn + 2k^2 - k^2)^2$
$= (n^2 + 3kn + k^2)(n^2 + 3kn + k^2)$
$= n^4 + 6kn^3 + 11k^2n^2 + 6k^3n + k^4$

RHS : $n(n+k)(n+2k)(n+3k) + k^4$
$= (n^2 + kn)(n+2k)(n+3k) + k^4$
$= (n^3 + 3kn^2 + 2k^2n)(n+3k) + k^4$
$= n^4 + 3kn^3 + 3kn^3 + 9k^2n^2 + 2k^2n^2 + 6k^3n + k^4$
$= n^4 + 6kn^3 + 11k^2n^2 + 6k^3n + k^4$

LHS = RHS, so must be true for all values of k and n.