Pythagoras for Tetrahedrons

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1 Diagram

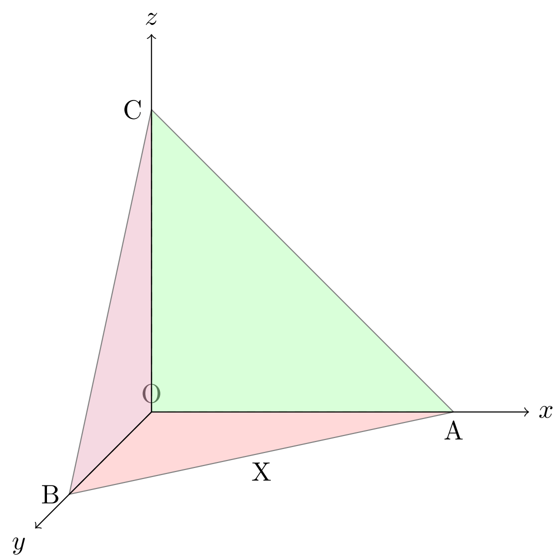

Figure 1: *Tetrahedron diagram, whereby the faces AOB, AOC, BOC and ABC are denoted as P, R, Q and S respectively*

2 Solution

Let there be a point X on AB such that AB is perpendicular to CX. Then we can write the following expressions for the area of AOB,

$$Area = \frac{(AO)(OB)}{2} \tag{1}$$

And also,

$$Area = \frac{(OX)(AB)}{2} \tag{2}$$

The length $(\mathrm{CX})^2$ can be expressed, using Pythagoras' Theorem, as the sum of the squared lengths of OX and CO,

$$(CX)^2 = (OX)^2 + (CO)^2 \tag{3}$$

which when rearranged gives,

$$(OX)^2 = (CX)^2 - (CO)^2 \tag{4}$$

Setting the two expressions for the area AOB equal to one another,

$$\frac{(AO)(OB)}{2} = \frac{(OX)(AB)}{2} \tag{5}$$

Substituting (4) into (5) and multiplying by 2 gives,

$$(AO)(OB) = \sqrt{(CX)^2 - (CO)^2}(AB)$$

$$\implies (AO)^2(OB)^2 + (CO)^2(AB)^2 = (AB)^2(CX)^2 \tag{6}$$

We can then use a similar process to write an expression for AB, using Pythagoras:

$$(AB)^2 = (OA)^2 + (OB)^2 \tag{7}$$

Substituting into (6),

$$(AO)^2(OB)^2 + (CO)^2[(OA)^2 + (OB)^2] = (AB)^2(CX)^2$$

$$(AO)^2(OB)^2 + (CO)^2(OA)^2 + (CO)^2(OB)^2 = (AB)^2(CX)^2 \tag{8}$$

Next we can write expressions for the area of each face of the tetrahedron,

$$S = \frac{(AB)(CX)}{2} \implies 4S^2 = (AB)^2(CX)^2 \tag{9}$$

$$P = \frac{(OA)(OB)}{2} \implies 4P^2 = (OA)^2(OB)^2 \tag{10}$$

$$Q = \frac{(OB)(OC)}{2} \implies 4Q^2 = (OB)^2(OC)^2 \tag{11}$$

$$R = \frac{(AO)(CO)}{2} \implies 4R^2 = (AO)^2(OC)^2 \tag{12}$$

By substituting equations (8), (9), (10) and (11) into (8),

$$4P^2 + 4R^2 + 4Q^2 = 4S^2 \implies P^2 + R^2 + Q^2 = S^2 \qquad \blacksquare$$

$\therefore$ in a tetrahedron with three perpendicular faces, that is P, R and Q, the square of the area of the final face is equivalent to the sum of the squared areas of the three perpendicular faces.