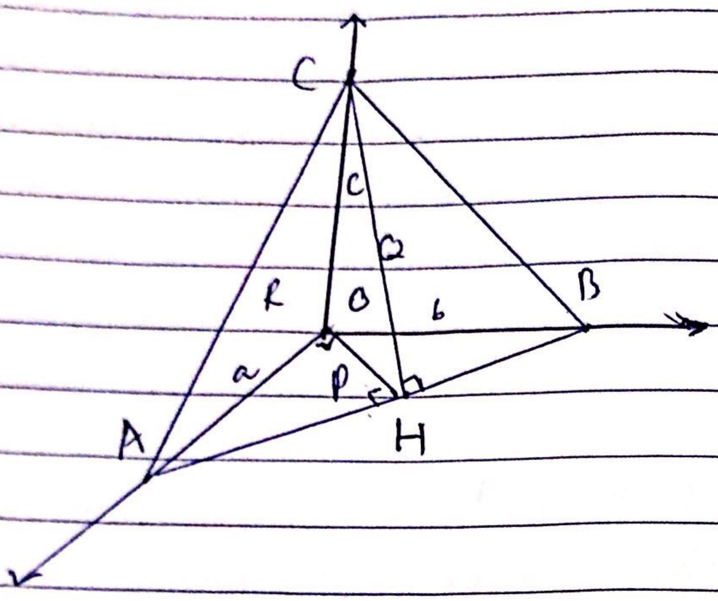
Pythagoras for a Tetrahedron



As OABC is a right-angled tetrahedron, ΔOAC, ΔOAB, and ΔOBC are all right-angled at O.

$$\Rightarrow \begin{cases} \text{Area}_{AOB} = P = \dfrac{ab}{2} \\ \text{Area}_{BOC} = Q = \dfrac{bc}{2} \\ \text{Area}_{COA} = R = \dfrac{ac}{2} \end{cases}$$

$$\Rightarrow LHS = P^2 + Q^2 + R^2 = \left(\frac{ab}{2}\right)^2 + \left(\frac{bc}{2}\right)^2 + \left(\frac{ac}{2}\right)^2 = \frac{a^2b^2 + b^2c^2 + a^2c^2}{4}$$

To calculate the area of the slanted face ABC, S, construct $CH \perp AB$ ($H \in AB$).

First, as ΔOAB is right-angled at O, $OA^2 + OB^2 = AB^2$ (Pythagoras)

$\Leftrightarrow a^2 + b^2 = AB^2$

$\Leftrightarrow AB = \sqrt{a^2+b^2}$

To calculate CH, the height of ΔABC, we must find OH to use Pythagoras' Theorem.

As ~~ΔOHA~~ $\text{Area}_{OHB} = \dfrac{ab}{2} = \dfrac{OH.AB}{2}$

$\Rightarrow ab = OH.AB$

$\Leftrightarrow ab = OH.\sqrt{a^2+b^2}$

Squaring both sides,

$a^2b^2 = OH^2(a^2+b^2)$

$\Leftrightarrow OH^2 = \dfrac{a^2b^2}{a^2+b^2}$

$\cancel{\Leftrightarrow OH = \sqrt{\dfrac{a^2b^2}{a^2+b^2}} = \dfrac{ab}{\sqrt{a^2+b^2}}}$

By applying Pythagoras' Theorem in ΔCOH right-angled at O,

$$OC^2 + OH^2 = CH^2$$

$$\Leftrightarrow c^2 + \dfrac{a^2b^2}{a^2+b^2} = CH^2$$

$$\Leftrightarrow CH = \sqrt{c^2 + \dfrac{a^2b^2}{a^2+b^2}}$$

Thus, $\text{Area}_{ABC} = S = \dfrac{CH \cdot AB}{2} = \dfrac{\sqrt{c^2+\dfrac{a^2b^2}{a^2+b^2}} \cdot \sqrt{a^2+b^2}}{2}$

$$\Leftrightarrow S^2 = \left(\dfrac{\sqrt{\left(c^2+\dfrac{a^2b^2}{a^2+b^2}\right)(a^2+b^2)}}{2}\right)^2$$

$$= \dfrac{\left(c^2+\dfrac{a^2b^2}{a^2+b^2}\right)(a^2+b^2)}{4}$$

$$= \dfrac{c^2(a^2+b^2) + a^2b^2}{4}$$

$\cancel{= \dfrac{}{a^2+b^2}}$

$$= \dfrac{a^2b^2+b^2c^2+a^2c^2}{4} = LHS$$

Thus, $P^2+Q^2+R^2 = S^2$