Investigation into Lewis Carroll's 'Sum of Squares of integers'. Carroll states that for two squares, say $x^2$ and $y^2$, there will be two squares which sum to the two previous squares multipled by two. ie $2(x^2 + y^2) = a^2 + b^2$. This has been proven previously by Hazel Pearson, who proved that

$$(x + y)^2 + (x - y)^2 = 2(x^2 + y^2)$$

This can be extended to the case where you have 3 squares multiplied by 3, there will be four other squares which sum to that number. This is demonstrated below

$$\begin{gathered}(x + y + z)^2 + (x - y)^2 + (x - z)^2 + (y - z)^2 \\ = (x^2+y^2+x^2+2xy+2xz+2yz)+(x^2-2xy+y^2)+(x^2-2xz+z^2)+(y^2-2yz+z^2) \\ = 3x^2 + 3y^2 + 3z^2 \\ = 3(x^2 + y^2 + z^2)\end{gathered}$$

Now we consider the case with n squares, say $a_1^2, a_2^2, \ldots, a_n^2$ multiplied by n equalling $x$ other squares.

$$\text{Consider } \left(\sum_{i=1}^{n} a_i\right)^2$$

$$\left(\sum_{i=1}^{n} a_i\right)^2 = 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} a_i a_j + \sum_{i=1}^{n} a_i^2$$

$$\text{Now consider } \sum_{r=1}^{n}\sum_{s=r+1}^{n} (a_r - a_s)^2$$

$$= \sum_{r=1}^{n}\sum_{s=r+1}^{n} (a_r^2 - 2a_r a_s + a_s^2)$$

$$= \sum_{r=1}^{n}\sum_{s=r+1}^{n} a_r^2 - 2\sum_{r=1}^{n}\sum_{s=r+1}^{n} a_r a_s + \sum_{r=1}^{n}\sum_{s=r+1}^{n} a_s^2$$

$$\text{Adding our first two functions } \left(\sum_{i=1}^{n} a_i\right)^2 + \sum_{r=1}^{n}\sum_{s=r+1}^{n} (a_r - a_s)^2$$

$$= \sum_{i=1}^{n} a_i^2 + \sum_{r=1}^{n}\sum_{s=r+1}^{n} a_r^2 + \sum_{r=1}^{n}\sum_{s=r+1}^{n} a_s^2$$

Having shown that the 'double' summation removes the extra terms in the form of $2a_ra_s$, I now need to show that

$$n\left(\sum_{r=1}^{n} a_r^2\right) = \sum_{i=1}^{n} a_i^2 + \sum_{r=1}^{n}\sum_{s=r+1}^{n} a_r^2 + \sum_{r=1}^{n}\sum_{s=r+1}^{n} a_s^2$$

Focusing at the moment on

$$\sum_{r=1}^{n}\sum_{s=r+1}^{n} a_r^2 + \sum_{r=1}^{n}\sum_{s=r+1}^{n} a_s^2$$

we can see that for $n = 1$ there will be $n - 1$ terms coming from the left summation and none from the right. Looking at $n = 2$ there will be $n - 2$ terms on the left, and there will be one term on the right, which is $n - 1$ in total. Now looking at the $n^{th}$ term we can see that there will be $n - n$ terms on the left and $n - 1$ on the left which is $(n - n) + (n - 1) = n - 1$ in total. Therefore, we can say that

$$\sum_{i=1}^{n} a_i^2 + \sum_{r=1}^{n}\sum_{s=r+1}^{n} a_r^2 + \sum_{r=1}^{n}\sum_{s=r+1}^{n} a_s^2$$

$$= \sum_{i=1}^{n} a_i^2 + (n-1)\sum_{r=1}^{n} a_r^2$$

$$= n\left(\sum_{r=1}^{n} a_r^2\right)$$

Which is what we were trying to prove earlier.

To finalise the proof, we need to know how many terms there will be which equal the n multiplied by n square numbers. To to this I have summed the number of bracketed pairs.

$$\sum_{r=1}^{n}\sum_{s=r+1}^{n} 1$$

$$= \sum_{r=1}^{n}(n - (r+1) + 1)$$

$$= \sum_{r=1}^{n}(n - r)$$

$$= \sum_{r=1}^{n} n - \sum_{r=1}^{n} r$$

$$= n^2 - \frac{n(n+1)}{2}$$

Since $T_{n-1} + T_n = n^2$, where $T_n$ is the $n^{th}$ triangle number, and $\frac{n(n+1)}{2}$ is the $n^{th}$ triangle number we, can see that

$$\sum_{r=1}^{n} \sum_{s=r+1}^{n} 1 = T_{n-1}$$

Therefore, there will be $T_{n-1}$ bracketed pairs, plus the one large square, which is $T_{n-1} + 1$ in total. Therefore, one can say that.

For n square numbers, multiplied by n, there will be $T_{n-1} + 1$ other square numbers which equal the same value

We can use this to determine how many 'terms' there will be in our final equation. We can see that there will be $T_{n-1} + 1$ terms, remembering to add the 'largest' term containing all of the other terms. This leads us to the conclusion that

$$n\left(\sum_{r=1}^{n} a_r^2\right) = \left(\sum_{i=1}^{n} a_i\right)^2 + \sum_{r=1}^{n} \sum_{s=r+1}^{n} (a_r - a_s)^2$$

which can be expressed in words as n square numbers, multiplied by n, will be equal to $T_{n-1} + 1$ other square numbers.