Square Difference - Ziwei Xu

1. Express each of the numbers 3, 5, 8, 12 and 16 as the difference of two non-zero square.

3: $2^2 - 1^2$

5: $3^2 - 2^2$

8: $3^2 - 1^2$ Trail and improvement

12: $4^2 - 2^2$

16: $5^2 - 3^2$

2. Prove that any odd number can be written as the difference of two square.

odd number = $2n + 1$

1: $1^2 - 0^2$

3: $2^2 - 1^2$

5: $3^2 - 2^2$

7: $4^2 - 3^2$

9: $5^2 - 4^2$

11: $6^2 - 5^2$

13: $7^2 - 6^2$

$2n - 1$: sequence of odd numbers

$n^2 - (n-1)^2$: also represents the sequence of odd numbers

$n^2 - (n^2 - 2n + 1) = 2n - 1$

2n is always even, even − 1 = odd

n can be any number, therefore all odd numbers can be represented by the difference of 2 consecutive square.

3. Prove that all numbers of the form 4k, k is a non-negative integer, can be written as the difference of 2 square.

4k : multiples of 4

4: $2^2 - 0^2$

8: $3^2 - 1^2$

12: $4^2 - 2^2$

16: $5^2 - 3^2$

20: $6^2 - 4^2$

This shows a trend of

$(k+1)^2 - (k-1)^2$

expression of this sequence is 4k:

it can also be expressed as $(k+1)^2 - (k-1)^2$

$= k^2 + 2k + 1 - (k^2 - 2k + 1)$

$= 4k$

k can be any integer, therefore all 4k can be represented by by the different of 2 squares

4. Prove that no number of the form $4k+2$, where k is a non-negative integer, can be written as the difference of two square.

$4k+2$: even, not multiple of 4

$x^2 - y^2$: when only one of x or y is odd, the difference is odd.

because odd × odd − even × even = odd
odd even

Therefore, for the difference to be even, x and y must both be even or odd.

When both odd: $(2n+1)^2 - (2y+1)^2$

$= 4n^2+4n+1 - (4y^2+4y+1)$

$= 4n^2 - 4y^2 + 4n - 4y$

$= 4(n^2 - y^2 + n - y)$ → This completely devisible by 4, which means it doesn't meet the conditions

When both even: $(2n)^2 - (2y)^2$

$= 4n^2 - 4y^2$

$= 4(n^2 - y^2)$ → Also completely divisible by 4, doesn't meet the conditions.

This proof that there's no possible combination of any type of integer which will result in a value of $4k+2$, because the values are either odd or divisible by 4.

First attempt of the question There's an attempt 2 — page 3 & 4 are investigations

5. pq, p & q are primes greater than 2, can be written as the difference of 2 square in exactly 2 distinct ways. Does this result hold if p is a prime greater than 2 and $q=2$?

primes greater than 2 are all odd.

odd · odd = odd

odd numbers can all be represented by the difference of 2 consecutive square.

$n^2 - (n-1)^2 = 2n - 1$

knowing these, pq can not only be represented by $n^2 - (n-1)^2$, but also:

$(k^2 - (k-1)^2)(y^2 - (y-1)^2) = (2k-1)(2y-1) = 4ky - 2y - 2k + 1$

because p & q can each be represented by the difference of 2 squares

Doesn't seem to give me anything......

$3 \times 5 = 15$ $\quad 4^2 - 1^2$ $\quad 8^2 - 7^2$

$3 \times 7 = 21$ $\quad 5^2 - 2^2$ $\quad 11^2 - 10^2$

$\quad 18^2 - 17^2$

$3 \times 11 = 33$ $\quad 7^2 - 4^2$

What about $3 \times 9 = 27$ $\quad 6^2 - 3^2$

→ Difference of 2 consecutive square

$n^2 - (n-1)^2 = 2n - 1$

$5 \times 7 = 35$ $\quad 6^2 - 1^2$

$5 \times 11 = 55$ $\quad 8^2 - 3^2$ $\quad$ ← From here I've noticed a pattern:

$5 \times 13 = 65$ $\quad 9^2 - 4^2$

pq, when $p < q$

pq can be expressed as a difference of 2 squares which has a difference of p.

e.g. $pq = a^2 - b^2$

$b = a - p$

a, increases by 1 everytime q increases by 2.

Q5 restart - This is my second attempt of this question. There's an attempt 3

pq, p & q are both prime greater then 2

odd · odd = odd → odd numbers can all be written as the difference of two consecutive squares.

$3\times5=15$ — 6 — 4^2-1^2 — increases every 2 terms — 8^2-7^2

$3\times7=21$ — 6 — 5^2-2^2 — 11^2-10^2

$\boxed{3\times9=27}$ — 6^2-3^2 → not prime. — 14^2-13^2

$3\times11=33$ — 6 — 7^2-4^2 — 18^2-17^2

Investigation:

For 15: 4^2-1^2 or 8^2-7^2

$(4-1)(4+1)$ or $(8-7)(8+7)$ → this matches with the factors of 15, so I took a brave guess

3×5 — 1×15

For 21: because it's a product of 1×21 or 3×7

it can be represented as

$(a+b)(a-b)$ where $\begin{cases} a+b=21 \\ a-b=1 \end{cases}$ or $\begin{cases} a+b=7 \\ a-b=3 \end{cases}$

$(10+11)(11-10)$ — $(5-2)(5+2)$

primes greater then 2 are all odd, odd + odd = even

there's always an integer solution to the $(a+b)(a-b)$

$a+b=x$

$a-b=y$

$2a=x+y$

$x+y$ must be even (true in this case)

However, it will only have 2 representation because it only has 2 pairs of factors, because it is a product of 2 primes.

Therefore, the product will have and only have 2 forms in the form of difference of 2 squares.

However, if $q=2$,

$pq=2p$ → this is a multiple of 2, and as mentioned in 4, unless p also has factor of 2, (the product is a multiple of 4), the product will not be able to be expressed as a difference of 2 square, however, there's no prime number which has a factor of 2 (all odd). This means if $q=2$ and p is a prime greater then 2, there won't be expressions.

Q5. Attempt 3

Let x be the product of pq, where p & q are prime greater then 2

$x = a^2 - b^2$ $\leftarrow$ Difference of 2 squares can be expressed as two factors.

$x = (a+b)(a-b)$

primes greater then 2 are all odd numbers

prime number only has factor of itself or 1

x can be expressed as : $p \times q$, $1 \times pq$ $\leftarrow$ these are the only two ways to factorise x

Now, would $p \times q$ and $1 \times pq$ fit the form $(a+b)(a-b)$?

$p = a+b$ $\qquad$ $pq = a+b$

$q = a-b$ $\qquad$ $1 = a-b$

Because all primes greater then 2 are odd

odd $= a+b$ $\qquad$ odd $\times$ odd $=$ odd $= a+b$

odd $= a-b$ $\qquad$ $1 = a-b$

a & b can be any integers, therefore all values can be represented.

odd + odd = even = $2a$ $\qquad$ odd + 1 = even = $2a$

Both solution seems to be true, therefore it works.

- We can conclude that there are two and only two solution because:
  - pq has and only has two distint way to be expressed as factors.
  - $a^2 - b^2 = (a+b)(a-b)$, which is two factors.

However, if $q = 2$,

$2p \rightarrow$ this is a multiple of 2, and as mentioned in Question 4, unless p also has factor of 2, (the product is a multiple of 4) the product will not be able to be expressed as a difference of 2 square, however, there's no prime number which has a factor of 2 (all odd). This means if $q = 2$ and p is a prime greater then 2, there won't be expressions.

Question 6.

$675 = 3^3 \times 5^2$

$a^2 - b^2 = (a+b)(a-b)$

675 has factors of:

$$\left.\begin{array}{l} 3 \times (3^2 \times 5^2) \\ 3^2 \times (3 \times 5^2) \\ 3^3 \times 5^2 \\ 5 \times (3^3 \times 5) \\ 1 \times (3^3 \times 5^2) \\ (3 \times 5) \times (3^2 \times 5) \end{array}\right\}$$ 6 pairs of factors

· This is all the combination of the prime factors, if we are looking for

· there can't be more as primes cannot be broken down further.

Or this could be determined by

$$\underbrace{3}_{\text{index of } 3} + \underbrace{2}_{\text{index of } 5} + \underbrace{3 \times 2}_{\text{combination of 3 and 5}} + 1 = \underline{12}$$

12 factors in total, therefore 6 pairs of distinct factors.

Kind of link into "proper factors", $m + n + mn - 1$ = number of proper factors.

this is 1 and the number itself

$m + n + mn - 1 + 2 = m + n + mn + 1$

Therefore, 675 has 6 distinct ways of being written as the difference of two squares, because:

- 675 can be expressed by 6 different pairs of factors.
- Each pair can be expressed as $(a+b)(a-b)$
- $(a+b)(a-b)$ is always a difference of 2 squares