Proper Factors — Ziwei Xu (Gilbert)

Question 1,

$3^2 \times 5^3$ have factors of:

3
3^2 } index of 3 : 2

5
5^2
5^3 } index of 5 : 3

$2+3+5=10$

There's no other factors it can have because prime factors cannot be factorised further, maximum possible numbers of combinations is also reached

$3^2 \times 5^3$ is not a proper factor

$3^2 \times 5$
3×5^2
$3^2 \times 5^2$
3×5^3
3×5 } combination of 3 and 5 $2 \times 3 - 1$

$3^m \times 5^n$

Number of proper factors is : $\underbrace{m}_{\text{index of 3}} + \underbrace{n}_{\text{index of 5}} + \underbrace{mn - 1}_{\text{combination of 3 and 5}}$

factorise

$(m+1)(n+1)-2$

$(m+1)(n+1)-2=10$

$(m+1)(n+1)=12$

$12 \begin{cases} 1 & 12 \\ 2 & 6 \\ 3 & 4 \end{cases}$

$2 \times 3 = 6$ combinations

$m \begin{cases} 0 \\ 1 \\ 2 \end{cases} \quad n \begin{cases} 11 \\ 5 \\ 3 \end{cases}$

or

$m \begin{cases} 11 \\ 5 \\ 3 \end{cases} \quad n \begin{cases} 0 \\ 1 \\ 2 \end{cases}$

6 integers of the form $3^m + 5^n$ have exactly 10 proper factors.

· Cannot be negative as a negative indice will lead to fraction.

· However, the question asks "how many other" therefore excluding $3^2 \times 5^3$,

$6 - 1 = 5$

5 other integers.

Question 2.

N, the smallest positive integer that has exactly 426 proper factors

we know, if $N = a^x \times b^y$, number of proper factor is $xy + x + y - 1$

or

$(x+1)(y+1) - 2$

what if $N = a^x \times b^y \times c^z$? (it has 3 prime factors)

e.g. $2^3 \times 3^4 \times 5^2$ has factors

$3 + 4 + 2 = 9 \rightarrow$ if only uses 1 prime factor ← number of possible combination

$3\times4 + 3\times2 + 4\times2 = 26 \rightarrow$ if using 2 prime factor

$3\times4\times2 - 1 = 23 \rightarrow$ if using 3 prime factors (excluding the number itself)

$9 + 26 + 23 = 58$

This can also be represented by:

$x + y + z + xy + xz + yz + xyz - 1$

↓ factorise

$(x+1)(y+1)(z+1) - 2$

Prove : x, y, z are number of possible combination when only using 1 prime factor

xy, xz, yz are number of possible combination when only using 2 prime factors

$xyz - 1$ is the number of possible combination when using all three prime factors.

From that, we can conclude for any positive integer, the number of proper factors can be expressed by its prime factors's indices.

Because this expression shows the total possible combination of all of its prime factors, and because primes only has factor of 1 or itself, there's no factor which is not included. All the proper factors are made up by 1 or more prime.

e.g. 1 prime factor $(x+1) - 2$

2 prime factors $(x+1)(y+1) - 2$

3 prime factors $(x+1)(y+1)(z+1) - 2$

4 prime factors $(x+1)(y+1)(z+1)(n+1) - 2$

. . . .

$426 = 2 \times 3 \times 71$, this can be represented as:

$$\left\{\begin{array}{l} 2\times3\times71 \\ 6\times71 \\ 2\times213 \\ 3\times142 \\ 426 \end{array}\right.$$

ignore

For N to be smallest, the prime factors has to be as small as possible.

This means :

if N has 1 prime factor :

$(x+1) - 2 = 426$

$x = 427$

smallest value is 2^{427}

if N has 2 prime factors :

$428 = 2^2 \times 107$, this can be represented as

$$\left\{\begin{array}{l} 2\times2\times107 \\ 4\times107 \\ 2\times214 \\ 428 \end{array}\right.$$

$(x+1)(y+1) - 2 = 426$

$(x+1)(y+1) = 428$

$x \begin{cases} 3 \\ 1 \end{cases}$ $y \begin{cases} 106 \\ 213 \end{cases}$ or $x \begin{cases} 106 \\ 213 \end{cases}$ $y \begin{cases} 3 \\ 1 \end{cases}$

values can be : $2^{106} \times 3^{3}$

$2^{213} \times 3$

$2^{3} \times 3^{106}$

2×3^{213}

: Smallest is $2^{106} \times 3^{3}$

if N has 3 prime factors :

$(x+1)(y+1)(z+1) - 2 = 426$

$(x+1)(y+1)(z+1) = 428$

I realised later I have made a mistake, I wrote 107 instead of 106

$x\begin{cases}1\end{cases} \quad y\begin{cases}1\end{cases} \quad z\begin{cases}106\end{cases}$

values can be : $2 \times 3 \times 5^{106}$

$2^{106} \times 3 \times 5$

$2 \times 3^{106} \times 5$

x and y share the same value, does the conclusion still stand true?

2 : 106	2, 3 : 106	2, 3, 5 : 105
3 : 1	2, 5 : 106	
5 : 1	3, 5 : 1	

add up to 426 still works

smallest value is $2^{106} \times 3 \times 5$

cannot have 4 prime factors because 428 only has 3 prime factors

$2 \times 2 \times 107$

2^{427} , $2^{106} \times 3^{3}$, $2^{106} \times 3 \times 5$

This is definitely the biggest

$3^3 = 27$

$2^{106} \times 3^{3}$

$2 \times 3 \times 5 = 30$

$2^{106} \times 2 \times 3 \times 5$

$2^{106} \times 3^{3}$ is smaller

Smallest value of N = $2^{106} \times 3^{3}$

$2^{106} \times 3^{3}$

$2^{106} \times 3 \times 5$

$3^3 = 27$

$3 \times 5 = 15$

$2^{106} \times 3 \times 5$ is smaller

Smallest value of N = $2^{106} \times 3 \times 5$