# Square Remainders

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Show that every odd square leaves remainder 1 when divided by 8, and that every even square leaves remainder 0 or 4.

Deduce that a number of the form $8n + 7$, where n is a positive integer, cannot be expressed as a sum of three squares.

Suppose an odd integer defined as $2k + 1$ where $k \in \mathbb{Z}$ then,

$$(2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$$

If $k$ is odd then $(k^2 + k)$ will be even since $k^2$ is odd and $k$ is odd so there addition must be even. If $k$ is even then $(k^2 + k)$ will be even since $k^2$ is even and $k$ is even so there addition must be even. In both cases $(k^2 + k)$ is even.

All even numbers are a multiple of 2 and their product with 4 will be a multiple of 8. Hence, the term $4(k^2 + k)$ will be a multiple of 8. This term will have a remainder of 0 when divided by 8. Therefore the remainder will always be 1.

Suppose an even integer defined as $2k$ then,

$$(2k)^2 = 4k^2$$

If $k$ is even then $k^2$ will be even. All even numbers are a multiple of 2 and their product with 4 will be a multiple of 8. Hence, the term $4(k^2)$ will be a multiple of 8 and the remainder will be 0 when dived by 8.

If $k$ is odd then $k^2$ will be odd. The odd result can be represented by $2m+1$ where $m \in \mathbb{N}$.

$$4(2m + 1) = 8m + 4$$

$8m$ has the remainder 0 when divided by 8. Therefore, 4 is the remainder.

It can be proved that $8n + 7$ can not be represented as the sum of 3 squares by using the aforementioned results. The sum of 3 squares can be represented four different cases and then individually proved to have a different remainder to $8n + 7$ when divided by 8. The remainder of $8n + 7$ is 7 since $8n$ has a remainder of 0.

The four cases of the sum of 3 squares is,

1. $(2k_1)^2 + (2k_2)^2 + (2k_3)^2$
2. $(2k_1)^2 + (2k_2)^2 + (2k_3 + 1)^2$
3. $(2k_1)^2 + (2k_2 + 1)^2 + (2k_3 + 1)^2$
4. $(2k_1 + 1)^2 + (2k_2 + 1)^2 + (2k_3 + 1)^2$

When divided by 8 no case has a remainder of 7. In case 1 each term can have either 0 or 4 as a remainder. The possible sums of the three remainders is 0, 8 and 12. Any remainder that is $\geq 8$ is divided again. This results in the possible remainders 0 and 4. Neither of which are 7 so can be eliminated as an option to represent $8n + 7$.

Following the same method as for case 1, where term $(2k + 1)^2$ always has a remainder of 1: case 2 has remainders 1 and 5, case 3 has remainders 2 and 6 and case 4 has a remainder of 1.

Therefore proved by exhaustion, $8m + 7$ cannot be expressed as a sum of three squares.