The Koch Snowflake

Sunday, December 11, 2022 12:49 PM

Here are some questions about the Koch Snowflake.

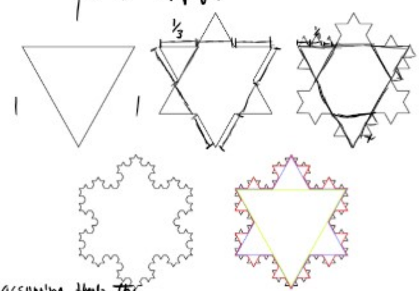
1. There are three edges in the first iteration of the snowflake. How many edges are there in the second iteration? How many in the third iteration?
2. Find a formula for the total number of edges of the n^{th} iteration.
3. If the first iteration has edges of length 1, how long are the edges in iteration 2?
4. Find a formula for the length of each edge in the n^{th} iteration.
5. Use your results to find a formula for the total length of the n^{th} iteration of the snowflake. What happens to the length of the snowflake as n gets larger?

Now consider the area of the Koch Snowflake.

1. Let the area of the first iteration be A (you might be able to actually calculate A given that you know the lengths of the edges). What is the extra area added on by each extra triangle in the second iteration? Show hint
2. How much area in total is added by the second iteration?
3. How much area is added by the third iteration?
4. Can you find an infinite sum for the area of the Koch Snowflake? Can you simplify this? Show hint

How do the length and the area behave as the number of sides increase?

Each iteration decreases 'the smallest triangle' length by multiplying by $\frac{1}{3}$



Note: I'm assuming that the iterations start from 1,2,3,etc, not 0,1,2,3,...

1) The second iteration has 12 edges while the third has 48. I found this out by simply counting.

2) I observed a pattern. The number of edges increase by multiplying 4 to the previous number of edges in the previous pattern

	Iterations 1st	2nd	3rd	nth
No. of edges	3	12	48	$4^{n-1}3$
	3×4^0	3×4^1	3×4^2	

$$4^{n-1}3$$

I saw that the exponent of 4 in each iteration was (the pattern number - 1)
Therefore, the number of edges in the n^{th} pattern must be $4^{n-1}3$

3) The length of the edges in the 2nd iteration is $\frac{1}{3}$.

4) By looking at the diagrams, we can see that the length of the edges continue to divide by 3.

	Iteration 1st	2nd	3rd	nth
Length of edges	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{3^{(n-1)}} \Rightarrow 3^{1-n}$
	$\left(\frac{1}{3}\right)^0$	$\left(\frac{1}{3}\right)^1$	$\left(\frac{1}{3}\right)^2$	$\left(\frac{1}{3}\right)^{n-1}$

$\frac{1}{a^n} = a^{-n}$

The length of each edge in the n^{th} iteration is $3^{(1-n)}$

5) To find the perimeter of the snowflake after each iteration, we find the product of the length of the edges and the no. of edges.

Length of edges $= 3^{(1-n)}$

No. of edges $= 4^{(n-1)}3$

$$\left(4^{n-1} \times 3\right) \times 3^{1-n}$$
$$= 4^{n-1} \times 3^1 \times 3^{1-n}$$
$$= 4^{n-1} \times 3^{2-n}$$
$$= 4^{n-1}3^{2-n}$$

$ab^m \times b^n = ab^{m+n}$

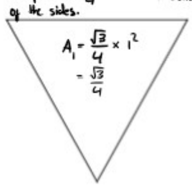
$(n=1)$	$(n=2)$	$(n=3)$	$(n=4)$
$4^{1-1}\times3^{2-1}$	$4^{2-1}3^{2-2}$	$4^{3-1}\times3^{2-3}$	$4^{4-1}\times3^{2-4}$
$4^0\times3^1$	$4^1\times3^0$	$4^2\times\frac{1}{3}$	$4^3\times\frac{1}{3^2}$
$\underline{3}$	$\underline{4}$	$\frac{16}{3}=\underline{5.\dot{3}}$	$\frac{64}{9}=\underline{\underline{7.\dot{1}}}$

Perimeter of snowflake $= 4^{n-1}3^{2-n}$

AREA

	Iteration 1st	2nd	3rd
Area	$\frac{\sqrt{3}}{4}$	$\frac{\sqrt{3}}{3}$	$\frac{10\sqrt{3}}{27}$

To find the area of an equivalent triangle, we can use the formula $\frac{\sqrt{3}}{4}s^2$ where s denotes the length of the sides.



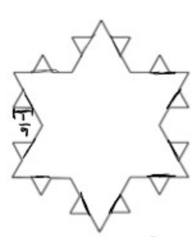
$$A_1 = \frac{\sqrt{3}}{4} \times 1^2 = \frac{\sqrt{3}}{4}$$

1) The area added by each extra triangle is $\frac{A_2 - A_1}{3}$, which is $\frac{\sqrt{3}}{36}$

$$\frac{A_2-A_1}{3} = \frac{\frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{4}}{3} = \frac{\frac{\sqrt{3}}{12}}{3} = \frac{\sqrt{3}}{36}$$

2) In the 2nd iteration, the area added from the 1st iteration is $A_2 - A_1$, which is $\frac{\sqrt{3}}{12}$

$$A_2 - A_1 = \frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{4}$$



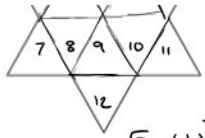
$$A_3 = A_2 + \frac{\sqrt{3}}{4} \times \left(\frac{1}{9}\right)^2 \times 3 \times 4$$
$$= \frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{4} \times \left(\frac{1}{9}\right)^2 \times 3 \times 4$$

2) In the 2nd iteration, the area added from the 1st iteration is $A_2 - A_1$, which is $\frac{\sqrt{3}}{12}$

$A_2 - A_1 = \frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{4}$

$= \frac{4\sqrt{3}}{12} - \frac{3\sqrt{3}}{12}$

$= \frac{\sqrt{3}}{12}$



$A_2 = A_1 + \frac{\sqrt{3}}{4} \times \left(\frac{1}{3}\right)^2 \times 3$

$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \times \frac{3}{9}$

$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}$

$= \frac{12\sqrt{3}}{36}$

$= \frac{\sqrt{3}}{3}$

$A_3 = A_2 + \frac{\sqrt{3}}{4} \times \left(\frac{1}{9}\right)^2 \times 3 \times 4$

$= \frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{4} \times \left(\frac{1}{9}\right)^2 \times 3 \times 4$

$= \frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{324} \times 12$

$= \frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{27}$

$= \frac{10\sqrt{3}}{27}$

3) The area added from the 2nd iteration to get the area of the third iteration is founded by $A_3 - A_2$, which is $\frac{\sqrt{3}}{27}$

$A_3 - A_2 = \frac{10\sqrt{3}}{27} - \frac{\sqrt{3}}{3}$

$= \frac{10\sqrt{3}}{27} - \frac{9\sqrt{3}}{27}$

$= \frac{\sqrt{3}}{27}$

4) There is a pattern between all the iterations:

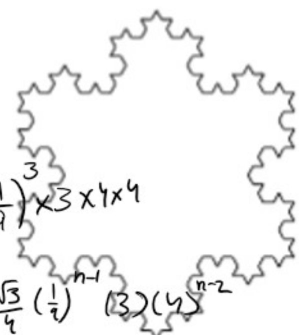
$A_1 = \frac{\sqrt{3}}{4}$

$\left(\frac{1}{3}\right)^2 = \frac{1}{9}$

$A_2 = \boxed{\frac{\sqrt{3}}{4}} + \frac{\sqrt{3}}{4} \times \left(\frac{1}{9}\right) \times 3$

$A_3 = \left[\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \times \left(\frac{1}{9}\right) \times 3\right] + \frac{\sqrt{3}}{4} \times \left(\frac{1}{9}\right)^2 \times 3 \times 4$

$A_4 = \left[\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\left(\frac{1}{9}\right) \times 3 + \frac{\sqrt{3}}{4} \times \left(\frac{1}{9}\right)^2 \times 3 \times 4\right] + \frac{\sqrt{3}}{4} \times \left(\frac{1}{9}\right)^3 \times 3 \times 4 \times 4$



$A_4 = A_3 + \frac{\sqrt{3}}{4} \times \left(\frac{1}{27}\right)^2 \times 3 \times 16$

$A_4 = \frac{10\sqrt{3}}{27} + \frac{\sqrt{3}}{4} \times \frac{48}{729}$

$= \frac{10\sqrt{3}}{27} + \frac{12\sqrt{3}}{729}$

$= \frac{270\sqrt{3} + 12\sqrt{3}}{729}$

$= \frac{282\sqrt{3}}{729}$

$= \frac{94\sqrt{3}}{729}$

In general:

$A_n = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\left(\frac{1}{9}\right)(3) + \frac{\sqrt{3}}{4}\left(\frac{1}{9}\right)^2(3)(4) + \cdots \frac{\sqrt{3}}{4}\left(\frac{1}{9}\right)^{n-1}(3)(4)^{n-2}$

$A_n = \frac{\sqrt{3}}{4}\left[1 + \left(\frac{1}{9}\right)(3) + \left(\frac{1}{9}\right)^2(3)(4) + \cdots \left(\frac{1}{9}\right)^{n-1}(3)(4)^{n-2}\right]$

$A_n = \frac{\sqrt{3}}{4}\left[1 + \frac{1}{3} + \frac{1}{3}\left(\frac{4}{9}\right) + \frac{1}{3}\left(\frac{4}{9}\right)^2 \cdots \frac{1}{3}\left(\frac{4}{9}\right)^{n-2}\right]$

∞

This expression is a geometric sequence which converges. The sum of the series is given by the formula:

$\text{sum} = \frac{a}{1-r}$

$a = \frac{1}{3}$

$r = \frac{4}{9}$

$S = \frac{\sqrt{3}}{4}\left[1 + \frac{\left(\frac{1}{3}\right)}{1 - \left(\frac{4}{9}\right)}\right]$

$S = \frac{\sqrt{3}}{4}\left[1 + \frac{\left(\frac{1}{3}\right)}{\left(\frac{5}{9}\right)}\right]$

$S = \frac{\sqrt{3}}{4}\left[1 + \frac{9}{15}\right]$

$S = \frac{\sqrt{3}}{4}\left[\frac{24}{15}\right]$

$S = \frac{\sqrt{3}}{4}\left[\frac{8}{5}\right]$

$= 0.692820\ldots$

≈ 0.693