To Swim or to Run? - Mahdi Raza

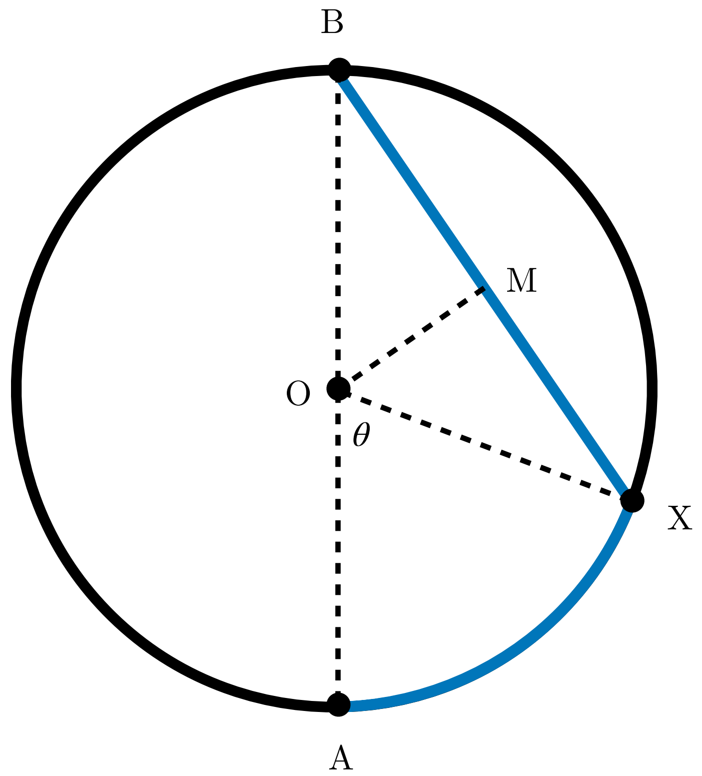
Swimming Speed: 1

Running Speed: k

Radius of pool: r

(i) Case 1 is covered in this diagram when $\theta = 0$

(iii) Case 3 is covered in this diagram when $\theta = \pi$



- **Running Distance:** arc AX which has length $\frac{\theta}{2\pi} \times 2\pi r = \theta r$
- **Running Time:** $\frac{\theta r}{k}$ because the running speed is k
- **Swimming Distance:** Chord XB which has length 2XM. $\angle \text{XOM} = 90 - \frac{\theta}{2}$

$$\sin\left(90 - \frac{\theta}{2}\right) = \frac{\text{XM}}{OX}$$

$$\text{XM} = r\cos\left(\frac{\theta}{2}\right)$$

$$\text{XB} = 2r\cos\left(\frac{\theta}{2}\right)$$

- **Swimming Time:** $\dfrac{2r\cos\left(\frac{\theta}{2}\right)}{1} = 2r\cos\left(\frac{\theta}{2}\right)$

So, total time, T = Running Time + Swimming Time

$$T = \frac{\theta r}{k} + 2r\cos\left(\frac{\theta}{2}\right)$$

To find the minimum time, $\frac{dT}{d\theta} = 0$ and $\frac{d^2T}{d\theta^2} > 0$

$$\frac{dT}{d\theta} = \frac{r}{k} - 2r\sin\left(\frac{\theta}{2}\right) \times \frac{1}{2}$$

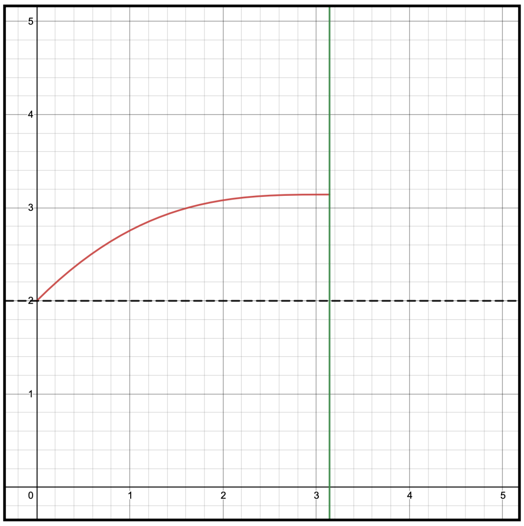
$$= r\left[\frac{1}{k} - \sin\left(\frac{\theta}{2}\right)\right]$$

Now the double derivative has to be positive for there to be a minima

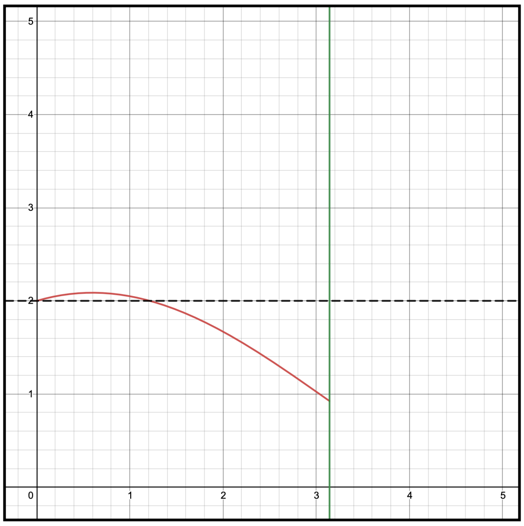
$$\frac{d^2T}{d\theta^2} = r\left[0 - \cos\left(\frac{\theta}{2}\right) \times \frac{1}{2}\right]$$

$$= \frac{-r}{2}\cos\left(\frac{\theta}{2}\right)$$

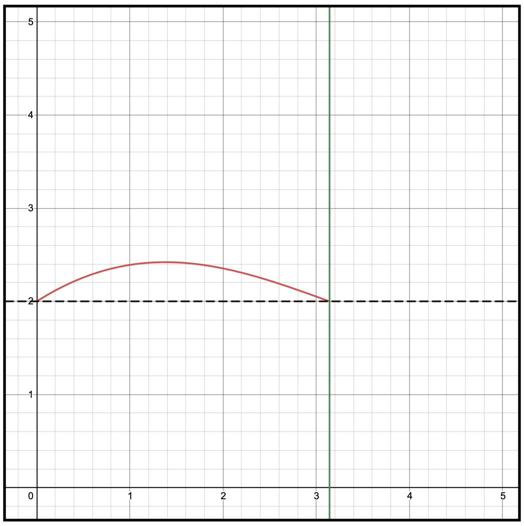
For $0 \le \theta \le \pi$, $\cos(\theta) \ge 0$ and so $\frac{d^2T}{d\theta^2} < 0$ which implies that there is no minimum time between $0 \le \theta \le \pi$. Hence, the minimum is at either of the end points. There are two cases for that, as shows in the two graphs below. r, the radius of pool just adds vertical stretch.



Ex: At $k = 1$, the minimum time is achieved at $\theta = 0$



Ex: At $k = 2$, the minimum time is achieved at $\theta = \pi$



There is a value of k for which the time taken at $\theta = 0$ is equal to the time taken at $\theta = \pi$. And the time taken is 2 for both. Let $r = 1$ since it just adds vertical stretch to the graph. Let's verify for $\theta = \pi$

$$2 = \frac{\theta r}{k} + 2r\cos\left(\frac{\theta}{2}\right).$$

$$2 = \frac{\pi}{k} + 2\cos\left(\frac{\pi}{2}\right).$$

$$2 = \frac{\pi}{k}$$

So we get $k = \frac{\pi}{2}$. We can thus conclude that:

For $0 < k < \frac{\pi}{2}$, the minimum time is achieved at $\theta = 0$. So option (i) swim directly.

For $\frac{\pi}{2} < k < \pi$, the minimum time is achieved at $\theta = \pi$. So option (iii) run around.