We have

$$0 \leq k \leq 9$$

$$10k - k^2 = -(k-5)^2 + 25 \leq 25$$

as $-(k-5)^2 \leq 0$.

and $-(k-5)^2 + 25 \geq 0$

as $|(k-5)| \leq 5$ for $0 \leq k$.

So $0 \leq 10k - k^2 \leq 25$

$(k-1), k, (k+1)$ are three consecutive numbers, so exactly one is divisible by 3.

So $3 \mid (k-1)k(k+1)$.

Let $N = 100a + 10b + c$, $0 \leq (a, b, c) \in \mathbb{Z} \leq 9$

and $1 \leq a$.

if $S = N$, we have

$$a + b^2 + c^3 = 100a + 10b + c$$

$$\Leftrightarrow c^3 - c - 99a = 10b - b^2$$

$$\underbrace{(c-1)c(c+1) - 99a}_{\text{divisible by } 3} = 10b - b^2$$

So $3 \mid 10b - b^2$. We also have

$0 \le 10b - b^2 \le 25$ as before from "u".

Considering the possible values:

$10b - b^2 = 3k$ for some $0 \le k \le 8$

$$\Leftrightarrow b^2 - 10b + 3k = 0$$

$$(b-5)^2 = 25 - 3k$$

b is an integer, so $25 - 3k$ must be a square. The only such values of k are

$k=0$ $k=3$, $k=7$, $k=8$

These give $b=9$, $b=7$, $b=6$ and ($b=10$ is
$b=4$ $b=3$. $b=1$ respectively. impossible)

We will consider these in turn:

$b = 9$:

$$c^3 - c - 99a = 9$$

So $(c-1)c(c+1) = 9(1+11a)$

3 divides only one of $(c-1), c, (c+1)$.

So 9 must divide $(c-1)c(c+1)$.

as $0 \le c \le 9$, $c = 8$

↳ $495 = 99a \Leftrightarrow a = 5 \in \mathbb{Z}$

or $c = 9$

↳ $711 = 99a \Leftrightarrow a = \frac{711}{99} \notin \mathbb{Z}$

50

504

9

$b = 7$:

$$c^3 - c - 99a = 21$$

$a \ge 1$, so we need $c^3 - c \ge 120$

$\Rightarrow c \ge 5$

$c = 5$

↳ $a = 1 \in \mathbb{Z}$

$c = 6$

↳ $a = \frac{189}{99} \notin \mathbb{Z}$

$c = 7$

↳ $a = \frac{315}{99} \notin \mathbb{Z}$

$c = 8$

$\hookrightarrow a = \frac{583}{99} \notin \mathbb{Z}$

$c = 9$

$\hookrightarrow a = \frac{694}{99} \notin \mathbb{Z}$

$b = 6$:

$$c^3 - c - 99a = 54$$

$$(c-1)c(c+1) = 9(11a+6) \equiv 0 \ [\text{mod } 9]$$

$c = 8$

$\hookrightarrow a = \frac{450}{99} \notin \mathbb{Z}$

$c = 9$

$\hookrightarrow a = \frac{666}{99} \notin \mathbb{Z}$

$b = 4$:

$$(c-1)c(c+1) = 24 + 99a \qquad c \geq 5$$

$$= 3(8 + 33a)$$

$c = 5$

$\hookrightarrow a = \frac{96}{99} \notin \mathbb{Z}$

$c = 6$

$\hookrightarrow a = \frac{156}{99} \notin \mathbb{Z}$

$c = 7$

$\hookrightarrow a = \frac{312}{99} \notin \mathbb{Z}$

$c = 8$

$\hookrightarrow a = \frac{480}{99} \notin \mathbb{Z}$

$c = 9$

$\hookrightarrow a = \frac{696}{99} \notin \mathbb{Z}$

$b = 3$:

$$(c-1)c(c+1) = 21 + 99a$$
$$= 3(7 + 11a)$$

which is in the case $b = 7$ gives $a = 1$ and $c = 5$.

$b=1$:

$$(c-1)c(c+1) = 99a+9$$
$$= 9(11a+1)$$

which gives as before for $b=9$
$a=5$ and $c=8$

This listing is exhaustive, so the only possible values for N are

518, 135, 175, 598.