

For some of these, I knew the answers straight away, but I wanted to prove it, so I went back to the rules of indices. Others were harder!

a)  $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

I know this works for all non negative integers: e.g.  $a = 4, b = 9$

LHS	RHS
$\sqrt{4} \times \sqrt{9}$	$\sqrt{4 \times 9}$
$= 2 \times 3$	$= \sqrt{36}$
$= 6$	$= 6$

$(a^m)^n = a^{mn}$  and  $a^{(1/2)} = \sqrt{a}, \therefore \sqrt{a} = a^{(1/2)}, \sqrt{b} = b^{(1/2)}, \sqrt{ab} = (ab)^{(1/2)}$

Rephrase the expression using indices and square throughout

LHS	RHS
$(a^{(1/2)} \times b^{(1/2)})^2$	$((ab)^{(1/2)})^2$
$= a^{(1/2) \times 2} \times b^{(1/2) \times 2}$	$= (ab)^{(1/2) \times 2}$
$= a^1 \times b^1$	$= (ab)^1$
$= (ab)^1$	

$\therefore$  any non-negative value for a, b will work

b)  $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$

This is the same as the first part, except that we are dividing instead of multiplying, so calculate:

$\frac{a \times a \times a \times a \times a \times a \times a \times \dots a}{b \times b \times b \times b \times b \times b \times b \times \dots b}$        $a$  appears  $m$  times in the numerator  
 $b$  appears  $m$  times in the denominator

$= \left(\frac{a}{b}\right) \times \left(\frac{a}{b}\right) \times \left(\frac{a}{b}\right) \times \left(\frac{a}{b}\right) \times \left(\frac{a}{b}\right) \times \left(\frac{a}{b}\right) \times \left(\frac{a}{b}\right) \times \dots \left(\frac{a}{b}\right)$        $\left(\frac{a}{b}\right)$  appears  $m$  times  $\therefore \left(\frac{a}{b}\right)^m$

E.g.  $\frac{4^3}{2^3} = \left(\frac{4}{2}\right) \times \left(\frac{4}{2}\right) \times \left(\frac{4}{2}\right) = \left(\frac{4}{2}\right)^3$

Using this we can say:  $\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$  and we know  $\sqrt{a} = a^{(1/2)}, \sqrt{b} = b^{(1/2)}$

$\therefore \frac{a^{(1/2)}}{b^{(1/2)}} = \left(\frac{a}{b}\right)^{(1/2)} \quad \therefore \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$

This will work for all non negative values for  $a$ , but  $b$  cannot be 0, because division by 0 is undefined.  
 $\therefore b$  can only be a positive integer.

Proof:  $\frac{100}{0} = x \quad \therefore 100 = 0x$  which is impossible: no multiple of 0 can equal 100 as  $0x = 0$ .

But:  $\frac{0}{100} = x \quad \therefore 0 = 100x$  which is possible when  $x = 0$ .

c) This has three layers of surds, so I will come back to it later.

d)  $a \times \sqrt{b} = \sqrt{ab}$

We are given:

From part (a) we know that  $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ , so substitute:  
 divide throughout by  $\sqrt{b}$

LHS	RHS
$a \times \sqrt{b}$	$\sqrt{ab}$
$a \times \sqrt{b}$	$\sqrt{a} \times \sqrt{b}$
$a$	$\sqrt{a}$

There are only two values for  $a$  that work, 0 and 1, because any power where 1 is the base produces 1. Similarly any power where 0 is the base produces 0. The answer is  $a = 0$ , or  $a = 1$ ,  $b$  can be any non negative value.

e)  $\frac{\sqrt{ab}}{\sqrt{a} + \sqrt{b}} = 1$   $\therefore$  in this question,  $\sqrt{ab} = \sqrt{a} + \sqrt{b}$

If  $a = 0$  then  $\frac{0}{\sqrt{b}} \neq 1$ , similarly if  $b = 0$  then  $\frac{0}{\sqrt{a}} \neq 1$ , because 0 divided by any number = 0

If  $a, b = 0$ , then  $\frac{0}{0} \neq 1$  because division by 0 is undefined.  $\therefore (a \neq 0), (b \neq 0), (a, b \neq 0)$

If  $a = b$ , then:

$$\frac{\sqrt{a^2}}{2 \times \sqrt{a}} = 1$$

divide throughout by  $\sqrt{a}$  to get:

$$\frac{\sqrt{a}}{2} = 1 \quad \therefore \sqrt{a} = 2 \quad \therefore a = 4$$

substitute  $a, b = 4$  back into the equation to get:

$$\frac{\sqrt{4 \times 4}}{\sqrt{4} + \sqrt{4}} = 1$$

$\therefore a = 4, b = 4$  satisfies the criteria, but this is unlikely to be the only solution.

To find any others I must come up with a rule. To do this I think I need to factorize, to find all solutions, but I spent ages trying to do so. Whenever I don't understand something, I wash some numbers through. I used square numbers to make it easier. I tried lots of examples, but these worked:

$$\frac{3^2}{2^2}, \frac{4^2}{3^2}, \frac{5^2}{4^2}, \text{etc ...}$$

These give:

Numerator:	Denominator
$\frac{\sqrt{3^2}}{\sqrt{2^2}} \times \frac{\sqrt{3^2}}{1} = \frac{3}{2} \times \frac{3}{1} = \frac{9}{2}$	$\frac{\sqrt{3^2}}{\sqrt{2^2}} + \frac{\sqrt{3^2}}{1} = \frac{3}{2} + \frac{6}{2} = \frac{9}{2}$
$\frac{\sqrt{4^2}}{\sqrt{3^2}} \times \frac{\sqrt{4^2}}{1} = \frac{4}{3} \times \frac{4}{1} = \frac{16}{3}$	$\frac{\sqrt{4^2}}{\sqrt{3^2}} + \frac{\sqrt{4^2}}{1} = \frac{4}{3} + \frac{12}{3} = \frac{16}{3}$
$\frac{\sqrt{5^2}}{\sqrt{4^2}} \times \frac{\sqrt{5^2}}{1} = \frac{5}{4} \times \frac{5}{1} = \frac{25}{4}$	$\frac{\sqrt{5^2}}{\sqrt{4^2}} + \frac{\sqrt{5^2}}{1} = \frac{5}{4} + \frac{20}{4} = \frac{25}{4}$

This series works, and I predict that it will continue to do so.

I can write this algebraically as:

$$\frac{\left(\frac{\sqrt{n^2}}{\sqrt{(n-1)^2}} \times \sqrt{n^2}\right)}{\left(\frac{\sqrt{n^2}}{\sqrt{(n-1)^2}} + \sqrt{n^2}\right)}$$

To prove that this works:

Numerator:

Denominator:

Start with:

$$\frac{\sqrt{n^2}}{\sqrt{(n-1)^2}} \times \sqrt{n^2}$$

$$\frac{\sqrt{n^2}}{\sqrt{(n-1)^2}} + \sqrt{n^2}$$

Then simplify:

$$= \frac{n}{(n-1)} \times \frac{n}{1}$$

$$= \frac{n}{(n-1)} + \frac{n}{1}$$

$$= \frac{n^2}{(n-1)}$$

$$= \frac{n}{(n-1)} + \frac{n^2 - n}{(n-1)}$$

$$= \frac{n^2}{(n-1)}$$

Numerator = denominator  $\therefore$  quotient must always be 1  $\therefore$  there will be infinite non-negative values for  $a, b$

But writing a general rule was a little trickier, and I went wrong! I only realised a few days later when I went back to rationalise the denominator of my original formula. So I went back to the numbers, and went through each line:

$a$  is  $\frac{9}{4}, \frac{16}{9}, \frac{25}{16}, \frac{36}{25}$  etc... which can be written as  $\frac{(m+2)^2}{(m+1)^2}$  where  $m$  is a positive integer

$b$  is 9, 16, 25, 36 etc... which can be written as  $(m+2)^2$  where  $m$  is a positive integer

Because both addition and multiplication are commutative  $a = (m+2)^2, b = \frac{(m+2)^2}{(m+1)^2}$  also works

$$\therefore \text{ we can say } \frac{\sqrt{ab}}{\sqrt{a} + \sqrt{b}} = \frac{\sqrt{\frac{(m+2)^2}{(m+1)^2} \times (m+2)^2}}{\sqrt{\frac{(m+2)^2}{(m+1)^2}} + \sqrt{(m+2)^2}} \quad \text{where } m \text{ is a positive integer}$$

Numerator

Denominator

If we substitute  $m = 5$ ,

$$\sqrt{\frac{(7)^2}{(6)^2} \times (7)^2}$$

$$\sqrt{\frac{(7)^2}{(6)^2}} + \sqrt{(7)^2}$$

Then simplify

$$= \frac{7}{6} \times 7$$

$$= \frac{7}{6} + \frac{7}{1}$$

$$= \frac{7}{6} + \frac{42}{6}$$

$$= \frac{49}{6}$$

$$= \frac{49}{6}$$

we can see that this works, and the rule will hold. This was actually quite difficult, and I had to have two goes at it to get it right!

$$f) \sqrt{a} - \sqrt{b} = \sqrt{a-b}$$

	LHS	RHS
We are given:	$\sqrt{a} - \sqrt{b}$	$\sqrt{a-b}$
Start by squaring both sides:	$a - (2 \times \sqrt{ab}) + b$	$a - b$
Subtract $a$ from both sides:	$b - (2 \times \sqrt{ab})$	$-b$
Rearrange:	$2b$	$2 \times \sqrt{ab}$
Simplify:	$b$	$\sqrt{ab}$
Square both sides to get:	$b^2$	$ab$
Divide both sides by $b$ :	$b$	$a$

For this equation to be true  $a = b$

Check:

Going back to the original equation, if  $a = b$ , then it will work BUT, there is another possibility also.

As any number minus zero remains unchanged  $a$  can be any non negative value when  $b = 0$ . However, when  $a = 0$ , the result contains the square root of a negative number, which is not allowed (we are looking for non negative values, not imaginary numbers).

g) has several layers, so I'll leave it until later.

$$h) \frac{\sqrt{a+b}}{\sqrt{c+d}} = (\sqrt{a} + b)(\sqrt{c} - d)$$

	LHS	RHS
We are given:	$\frac{\sqrt{a} + b}{\sqrt{c} + d}$	$(\sqrt{a} + b)(\sqrt{c} - d)$
Rearrange:	$\frac{\sqrt{a} + b}{\sqrt{a} + b}$	$(\sqrt{c} + d)(\sqrt{c} - d)$
Simplify:	1	$c - d^2$
Rearrange:	$d^2 + 1$	$c$

**In the original statement**  $a, b = 0$  is permitted as both sides of the equation would be 0, but  $c, d = 0$  is not permitted, as division by 0 is undefined.

$\therefore$  the original statement is true when either  $c = d^2 + 1$  or  $a, b = 0$  (but  $c, d \neq 0$ )

If you substitute values in for  $d$ , it becomes clear: e.g. if  $d = 5, c = 26$

$$\frac{\sqrt{a} + b}{\sqrt{26} + 5} = (\sqrt{a} + b)(\sqrt{26} - 5) \quad \text{becomes} \quad \frac{\sqrt{a} + b}{\sqrt{a} + b} = (\sqrt{26} + 5)(\sqrt{26} - 5)$$

then it doesn't matter what non negative values you have for  $a, b$  as the RHS in the **rearranged statement** produces 1 (a clever twist on the difference of squares)

But if  $a, b = 0$  **in the original statement**, then both sides of the statement become 0.

$$i) \sqrt{5 + (2 \times \sqrt{6})} = \sqrt{a} + \sqrt{b}$$

	LHS	RHS
We are given:	$\sqrt{5 + (2 \times \sqrt{6})}$	$\sqrt{a} + \sqrt{b}$
Square each side:	$5 + (2 \times \sqrt{6})$	$a + (2 \times \sqrt{ab}) + b$
Rearrange:	$5 + (2 \times \sqrt{6})$	$(a + b) + (2 \times \sqrt{ab})$

It's logical to suppose that  $(a + b) = 5$ , and  $ab = 6$  Factors of 6 are (1x6) (2x3) Numbers which sum to 5 are (0+5) (1+4) (2+3) 2,3 are the only numbers that appear in both sets,  $\therefore \{a, b\} = \{2, 3\}$  in any order.

To check: substitute:

	LHS	RHS
	$\sqrt{5 + (2 \times \sqrt{6})}$	$\sqrt{2} + \sqrt{3}$
Square each side:	$5 + (2 \times \sqrt{6})$	$2 + (2 \times \sqrt{6}) + 3$
Rearrange:	$5 + (2 \times \sqrt{6})$	$5 + (2 \times \sqrt{6})$

$$g) \sqrt{a} + \sqrt{b} = \sqrt{a + b + \sqrt{4ab}}$$

	LHS	RHS
We are given:	$\sqrt{a} + \sqrt{b}$	$\sqrt{a + b + \sqrt{4ab}}$
Square each side:	$a + (2 \times \sqrt{ab}) + b$	$a + b + \sqrt{4ab}$
Subtract $a + b$ from each side	$2 \times \sqrt{ab}$	$\sqrt{4ab}$
Simplify	$2 \times \sqrt{ab}$	$2 \times \sqrt{ab}$

$\therefore a, b$  can be any non-negative value for the equation to work!

$$c) \sqrt{23 - (6 \times \sqrt{6 - (4 \times \sqrt{2})})} = \sqrt{a} + \sqrt{b}$$

This was quite difficult and I had to walk away for a while.

The numbers on the LHS simplify to  $3 + \sqrt{2} \therefore \sqrt{a} + \sqrt{b} = \sqrt{9} + \sqrt{2}$ , or vice versa.

But this isn't a solution, so I worked backwards to try and unravel the algebra and prove the result:

LHS

RHS

We are given:

$$\sqrt{23 - \left(6 \times \sqrt{6 - (4 \times \sqrt{2})}\right)}$$

$$\sqrt{a} + \sqrt{b}$$

Square both sides:

$$23 - \left(6 \times \sqrt{6 - (4 \times \sqrt{2})}\right)$$

$$(\sqrt{a} + \sqrt{b})^2$$

Rearrange:

$$23 - (\sqrt{a} + \sqrt{b})^2$$

$$\left(6 \times \sqrt{6 - (4 \times \sqrt{2})}\right)$$

Divide throughout by 6:

$$\frac{23 - (\sqrt{a} + \sqrt{b})^2}{6}$$

$$\sqrt{6 - (4 \times \sqrt{2})}$$

Square both sides

$$\left(\frac{23 - (\sqrt{a} + \sqrt{b})^2}{6}\right)^2$$

$$6 - (4 \times \sqrt{2})$$

Rearrange:

$$4 \times \sqrt{2}$$

$$6 - \left(\frac{23 - (\sqrt{a} + \sqrt{b})^2}{6}\right)^2$$

Divide throughout by 4:

$$\sqrt{2}$$

$$\frac{6 - \left(\frac{23 - (\sqrt{a} + \sqrt{b})^2}{6}\right)^2}{4}$$

Add  $\sqrt{9}$  to both sides:

$$\sqrt{9} + \sqrt{2}$$

$$\sqrt{9} + \left(\frac{6 - \left(\frac{23 - (\sqrt{a} + \sqrt{b})^2}{6}\right)^2}{4}\right)$$

Now insert  $a = 9, b = 2$   
into RHS and simplify:

$$= 3 + \sqrt{2}$$

$$3 + \left(\frac{6 - \left(\frac{23 - (\sqrt{9} + \sqrt{2})^2}{6}\right)^2}{4}\right)$$

NOTE:

$$(\sqrt{9} + \sqrt{2})^2 = 11 + (6 \times \sqrt{2})$$

$$= 3 + \left(\frac{6 - (2 + \sqrt{2})^2}{4}\right)$$

$$(2 + \sqrt{2})^2 = 6 + (4 \times \sqrt{2})$$

$$= 3 + \left(\frac{4 \times \sqrt{2}}{4}\right)$$

$$= 3 + \sqrt{2}$$

Phew! This last question was really hard!

Also, as addition is commutative,  $b = 9, a = 2$  is also possible. Brilliant question!