15/01/16 Weida

Picture the Process I

Temperature of a cup of tea over time.

We assume that the rate at which the temperature decreases is proportional to the temperature at that point.
If T represents temperature and t represents time, then $-\frac{dT}{dt} \propto T$
This means, approximately, that the higher the temperature, the faster the temperature decreases. On a graph of T against t, for lower values of t, the gradient would be steeper.
From this we may set up a differential equation and solve it to find temperature as a function of time.

$-\frac{dT}{dt} = kT$, where k is some positive constant.

Separating the variables, we have

$$\frac{1}{T}\frac{dT}{dt} = -k$$

Integrating with respect to t,

$$\int \frac{1}{T}\frac{dT}{dt}\,dt = \int -k\,dt$$

$$\int \frac{1}{T}\,dT = \int -k\,dt$$

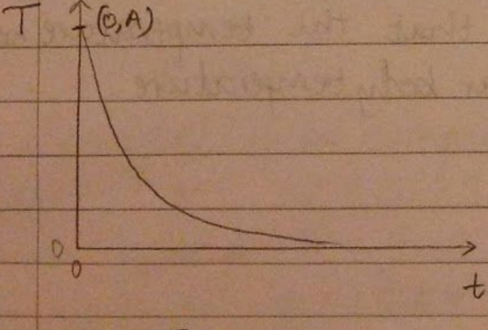
$$\ln T = -kt + c$$

Changing to exponential notation

$$T = e^{-kt+c}$$

$$= e^{-kt} \cdot e^{c}$$

$$= Ae^{-kt}$$ where A is some constant



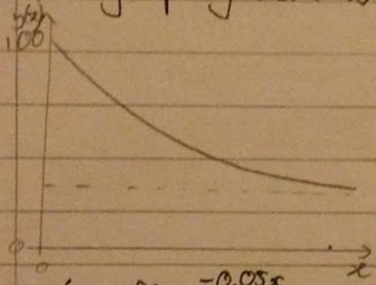
Left is the graph of $T = Ae^{-kt}$ for $t \geq 0$ (as we generally only consider positive time values - although ~~that~~ it is arbitrary where we set $t=0$. Here we are saying that $t=0$ where the cup of tea is ready).

The T-intercept (i.e. at $t=0$) is $T = Ae^{0} = A$ $(0, A)$.
This is the starting temperature of the tea.

Also, the t-axis is an asymptote to the graph. This means that as $t \to \infty$, $T \to 0$.

On this graph, $T=0$ ~~would~~ represents room temperature (so the units for temperature are unlikely to be °C on this graph!). As time goes on, the temperature of the cup of tea decreases at a decreasing rate, proportional to the temperature gradient (the difference in temperature of the tea and the room). As heat is transferred from the tea (which we have assumed is hotter than the room) to the room, the temperature of the tea approaches that of the room. (However, in real life, the tea would eventually reach room temperature)

The graph given is $y(x) = 20 + 80e^{-0.05x}$



This is similar to $T = Ae^{-kt}$ but has ~~an asymptote~~ a y-intercept of $y = 20 + 80e^{-0.05\times 0}$
$= 20 + 80 = 100$ instead of the coefficient of $e^{-0.05x}$, 80, and an asymptote of $y = 20$ ($80e^{-0.05x} \to 0$ as $x \to \infty$) instead of $T = 0$.

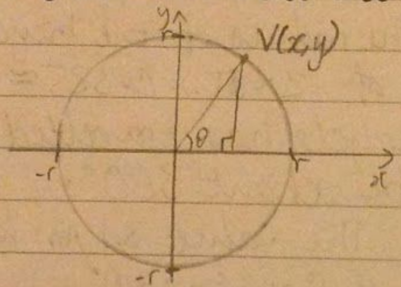
This is simply down to choice of units. $y(x)$ probably uses °C for temperature (rather than a scale with $T=0$ at room temperature) implying that the tea starts at 100°C and ends up eventually at 20°C.

It is likely that the unit for x is minutes, because after 30 minutes it is likely that the temperature would be $y(x) = 20 + 80e^{-0.05\times 30} \approx 38$°C, near body temperature.

Weida

<u>Height of the valve on a bicycle tyre as the bicycle moves forwards</u>

We assume that the wheel rotates at a constant rate,



and that the bicycle is travelling horizontally. Let $V(x,y)$ be the position of the valve. As the bicycle travels forwards, if we ignore the linear motion of the valve in the direction that the bicycle is travelling, the valve is essentially tracing a circle. (See graph sketch above.)

Consider the graph above. When V has turned through an angle of θ anticlockwise from the x-axis, looking at the right-angled triangle drawn, $y = r\sin\theta$.

Thinking back to the bicycle, y is the height of the valve above the height of the centre of the tyre, and r is the radius of the tyre.

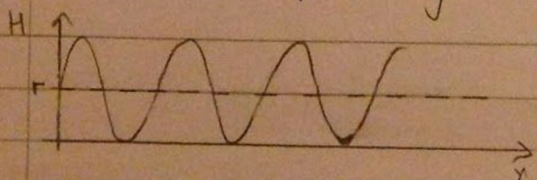
~~If we are consi~~ Also, as the ~~angular~~ speed is constant, we can say that $\theta = kx$, where k is some constant and x is the distance that the bicycle has travelled – ie. the angle through which the valve has turned is proportional to the distance travelled.

As well as this, we may wish to convert the height above the centre of the tyre y to the height of the valve above the ground H, by adding r. *

Combining this, we have $H = r\sin kx + r$

[* r is the height of the centre of the tyre above the ground.]

This assumes that the valve begins at the same height as the centre of the tyre.



The given graph is $y(x) = 0.325(\sin 3.1x + 1)$ or $y(x) = 0.325\sin 3.1x + 0.325$. Comparing with $H = r\sin kx + r$, we see that $r = 0.325$. As this is the radius of the tyre, the unit is likely to be metres.

We also see that $k = 3.1$. From $\theta = kx$, we see that k is the angle that the valve turns through per unit distance (metre) travelled by the bicycle. The radius of the bicycle is 0.325m, giving a circumference of $2 \times \pi \times 0.325 \approx 2$m. This means that when the bicycle has travelled 2m ($x = 2$), the angle should be about $360°$:

$360° \approx 2k$, so $k \approx 180°$. The value 3.1 for k suggests units of radians, as $3.1^c \approx \pi^c = 180°$.

Radius of a spherical balloon as it is inflated.

We assume that the balloon is being inflated at a constant rate ~~Let~~ – i.e. the volume of the balloon increases steadily with time.

Let V be the volume of the balloon, t the time elapsed since we started inflating the balloon and r the radius of the balloon.

From the assumption, $V \propto t$

Also, we know that the volume of ~~that~~ a sphere is given by $V = \frac{4}{3}\pi r^3$, so $V \propto r^3$

Therefore, $t \propto r^3$, or $t = kr^3$ for some constant k.

Rearranging to make r the subject, we have

$$r = \sqrt[3]{\tfrac{t}{k}}$$
$$= \sqrt[3]{\tfrac{1}{k}} \cdot \sqrt[3]{t}$$
$$= A\sqrt[3]{t}$$

The graph of this looks like $y = x^3$, but '~~sideways~~' reflected in $y = x$. The graph given is $y(x) = 0.6204\sqrt[3]{x}$.

Let us consider what the constant A represents.

In $V \propto t$, the constant of proportionality is the volume of air put into the balloon per unit time.

Let's say $V = pt$.

In $V = \frac{4}{3}\pi r^3$, the constant of proportionality is $\frac{4}{3}\pi$.

So $V = pt = \frac{4}{3}\pi r^3$

$\therefore \quad t = \frac{4}{3p}\pi r^3$

$\therefore \quad k = \frac{4}{3p}\pi$, where p is the volume of air put into the balloon per unit time.

$A = \sqrt[3]{\tfrac{1}{k}}$

In $y(x)$, if we compare to $r = A\sqrt[3]{t}$, $A = 0.6204 = \sqrt[3]{\tfrac{1}{k}}$

so $\frac{1}{k} = 0.23878\ldots$

and $k = 4.1877\ldots = \frac{4}{3p}\pi$

so $p = \frac{4}{3 \times 4.18\ldots}\pi = 1.000239\ldots = 1.00$ (3sf)

After 4 units of time on the graph $y(x)$, radius = 1 unit.

Perhaps the units of time are seconds, and 1 unit = 5cm for the radius, suggesting that every second, $1 \times 5^3 = 125 cm^3$ of air was put into the balloon.