Hollow Squares

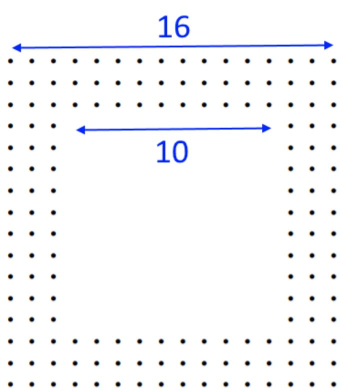
Vid Kavčič
high school Srednja šola Črnomelj
Slovenia

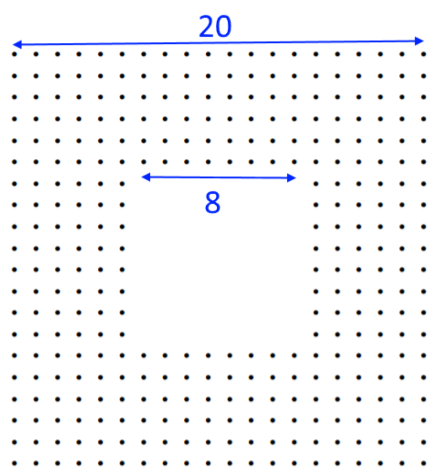
1 Introduction – the problem

A hollow square was a popular formation for an infantry battalion designed to cope with Cavalry charges in Napoleonic battles.

For example, the picture on the right represents a recreation of Wellington's army at Waterloo.

There are two diagrams below showing symmetrical hollow square formations.





2 Explaining the two methods for counting

Let a be the length the outer square side and b the length of the internal (hollow) square side.

2.1 Alison's method

In general, we are able to express the number of soldiers using Alison's method, and simplify that expression.

$$4 \cdot \left(\frac{a-b}{2} + b \right) \cdot \left(\frac{a-b}{2} \right) = (a - b + 2b) \cdot (a - b) = (a + b) \cdot (a - b)$$

2.2 Charlie's method

Similarly, we are able to express the number of soldiers by Charlie's method, which should be the same in Allison's case, and simplify that expression.

$$a^2 - b^2 = (a - b) \cdot (a + b),$$

where we used the theorem for factorizing the difference of two squares.

As seen from the example, both approaches are useful. Therefore, if N is number of soldiers, then the equation below follows.

$$N = (a + b) \cdot (a - b) \tag{1}$$

3 The problem of 960 soldiers

A general has 960 soldiers. In how many different ways can he arrange his battalion to form a symmetric hollow square?

According to the equation (1) the statement

$$960 = (a + b) \cdot (a - b)$$

must be true. Therefore, it is necessary to find all possible pairs (x, y) of natural numbers $x = a + b$ and $y = a - b$ (it obviously holds that $x > y$) such as $x \cdot y = 960$. Here are the options.

x	y
1	960
2	480
3	240
4	192
5	160
6	120
10	96
12	80
15	64
16	60
20	48
24	40
30	32

Table 1: All possible pairs $(x, y), x > y$ so that $x \cdot y = 960$

There is a procedure which enables the reduction of the number of possibilities at the very beginning.

Deliberation: In the follow up, we are going to solve the system of two equations; $a + b = x$ and $a - b = y$. We can combine these two equations to get $x + y = 2b$ or equivalently $b = \frac{x+y}{2}$, which satisfies that $b \in \mathbb{N}$.

Therefore the sum $x + y$ must be even so it is divisible by number 2. And so numbers x and y must both be even or odd. We are now able to eliminate some of the possible products. The table 2 represents the remaining ones.

We also include the values of a and b into the table, because we can get the values by solving the system of linear equations.

At the end, we can conclude that there are 10 different possibilities for formatting the "*hollow square*". ★

x	y	a	b
2	480	241	239
4	192	98	94
6	120	63	57
10	96	53	43
12	80	46	34
16	60	38	22
20	48	34	14
24	40	32	8
30	32	31	1

Table 2: All possible pairs $(x, y), x > y$ so that $x \cdot y = 960$, second version

4 General strategy

Can you find a general strategy for arranging any possible battalion into all possible symmetric hollow squares?

This strategy is similar to the way we have solved the definite problem.

In case of N soldiers in the battalion, the procedure is the following:

- Make a table of all possible products of two natural numbers x, y such that $x \cdot y = N$.
- If it is not true that both x and y are **odd** or both x and y are **even**, then eliminate the possible pair.
- Solve the system of linear equations $a + b = x$ and $a - b = y$ to get a and b, which enable you to form the shape of hollow square of the battalion.

NOTE: The method remains the same for the **non symmetric** *hollow squares*, because the number of soldiers is crucial for the problem. Therefore, the empty square could be placed anywhere (just not on the outer square borders, of course). ★