# Multiple Surprises

This question is to do with consecutive numbers and multiples. Throughout this solution I will use $a$ as the variable and represent said consecutive numbers by $a + 1$, $a + 2$.... (that is, when I need to).

Now, let's get onto the 1st question:

## Can you find **three consecutive numbers** where the first is a multiple of 2, the second is a multiple of 3 and the third is a multiple of 4?

One trivial solution to this is 2, 3 and 4. Another one, also not that hard to find, is 14, 15 and 16. Notices the numbers in each set have a difference of 12 from one set to another. What is going on? Well, it just so happens that **12** is the **L**owest **C**ommon **M**ultiple (or **LCM**) of 2, 3 and 4. What happens is that we start with what I'm going to call the 'trivial set', **TS** for short. In this case it is 2, 3 and 4. If we add 12 each of these numbers, we get another set. Why? Because when we add 12 to 2, we are adding two numbers which are both multiples of **2** together, and so we must end up with a multiple of **2.** This is in fact something I will extensively refer to later on, so I will just present it here:

***'Multiple Theorem'***

If we have two numbers $a$ and $b$, where $a = xy$ and $b = zy$, then $a + b = y(x + z)$, and so $a + b$ is a multiple of $y$.

Using this 'theorem', we can see that since 12 is a factor of 3 and a factor of 4, when we add 12 to 3 and 12 to 4, the resulting numbers will be multiples of 3 and 4, respectively. Since 12 is the **LCM** of these numbers, it will take us from the **TS** to the next smallest set. By repeating the process, we will get an infinity of sets.

Now, let's get on to the rest of the questions.

## What if the first is a multiple of 3, the second is a multiple of 4 and the third is a multiple of 5?

Well, the **TS** in this example is 3, 4 and 5. The **LCM** of 3, 4 and 5 is 3*4*5 (they have no common factor): that is 60. So the next smallest set is 63, 64 and 65. Sure enough, 63 = 3*21, 64 = 4*16, and 65 = 5*13. If we keep adding 60, we get more sets.

## What if the first is a multiple of 4, the second is a multiple of 5, and the third is a multiple of 6?

Well, the **TS** in this example is 4, 5 and 6. The way to find the **LCM** of 4, 5 and 6 is to find the find the **LCM** of 4 and 5, then the **LCM** of the **LCM** of 4 and 5 with 6. The **LCM** of 4 and 5 is 20. The **LCM** of 20 and 6 is 60. So the next smallest set should be 64, 65 and 66. We've already established previously that 64 and 65 are multiples of 4 and 5, respectively. Also, it shouldn't be hard to see that 66 = 6*11. So if we keep adding 60, we get other sets.

## Is there a way to find sets of **four consecutive numbers** which are multiples of 2, 3, 4 and 5 (in this order)?

Well, we've already established that the **LCM** of 2, 3 and 4 is 12. Since 12 and 5 have no common factor, the **LCM** of 2, 3, 4 and 5 is 60. (60, by the way, is a multiple of many numbers). The **TS** in this example is 2, 3, 4 and 5. So the next smallest set is 62, 63, 64 and 65. 62 = 2*31, and we've already established that 63, 64 and 65 are multiples of 3, 4 and 5, respectively. Here, if we keep adding 60, we get the next sets that meet the conditions.

## Or **five consecutive numbers** which are multiples of 2, 3, 4, 5 and 6 (in this order)?

Well, we already know the **LCM** of 2, 3, 4 and 5 is 60. Furthermore, 60 = 6*10. So the **LCM** of all the numbers presented is 60. The **TS** in this case is 2, 3, 4, 5 and 6. The next smallest set, therefore, should be 62, 63, 64, 65 and 66. We have already established that the numbers are multiples of 2, 3, 4, 5 and 6, respectively, so this works!

## Can we generalize this to **n consecutive numbers** which are multiples of ***k*, *k* + 1, *k* + 2… *k* + *n* − 1** (in this order)?

We sure can. Throughout this solution, we have needed little to no algebra (only for the 'Multiple Theorem'). Now, however, we will need some algebra, but only a little. The method is this:

1. Find the **TS**. In this case it will be $k$, $k + 1$… $k + n - 1$.
2. Next, find the **LCM**. This can be done with the method I described in the answer to the 3rd question. Let this be $l$.
3. The next smallest set will be $k + l$, $k + l + 1$…, $k + n + l - 1$. To find other sets, just add $2l$, $3l$ and so on. To find the $r^{th}$ set, just add $l(r - 1)$ to each number in the **TS**.

The solution to the question is now complete.