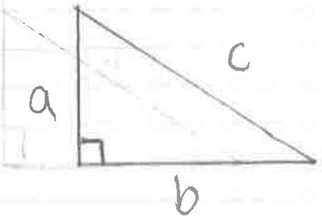
Pythagoras Perimeters - nrich

1. If this right angled triangle has a perimeter of 12 units, it's possible to show that the area $= 36 - 6c$ square units. Can you prove this?



$$\text{Perimeter} = a+b+c$$
$$= 12 \text{ units}$$

$$\text{Area} = \tfrac{1}{2}ab$$
$$= 36 - 6c \text{ units}^2$$

$$a+b+c = 12$$
$$a+b = 12 - c$$

(The c needs to be moved over for when you divide ab by 2 so then you can find the area.)

Squaring both sides as:

$a^2 + b^2 = c^2$ (Pythagoras' theorem)

$$\Rightarrow (a+b)^2 = (12-c)^2$$
$$(a+b)(a+b) = (12-c)(12-c)$$
$$a^2 + 2ab + b^2 = 144 - 24c + c^2$$

From this, you can cancel out $a^2+b^2=c^2$ as it equals the same amount.

$$\Rightarrow 2ab = 144 - 24c \quad)\div 2$$
$$ab = 72 - 12c \quad)\div 2 \text{ (to get the area)}$$
$$\frac{ab}{2} = 36 - 6c$$

$\therefore$ Area $= 36 - 6c$ - You can check this theory because the triangle with the perimeter of 12 is also a pythagorean triple (3, 4, 5).

$$\frac{ab}{2} = 36 - 6c$$
$$\frac{(3 \times 4)}{2} = 36 - (6 \times 5)$$
$$\frac{12}{2} = 36 - 30$$
$$6 = 6$$

$\therefore$ $36 - 6c = \frac{ab}{2}$ (the area) in a triangle with perimeter $= 12$.

2. Can you adapt your method to prove that a triangle with the perimeter of 30 units has $225 - 15c$ units squared as its area.

$$\text{Perimeter} = a+b+c$$
$$= 30 \text{ units}$$

$$\text{Area} = \tfrac{1}{2}ab$$
$$= 225 - 15 \text{ units}^2$$

$$a+b+c = 30$$
$$a+b = 30 - c \quad \text{(to get ab seperate)}$$

Square both sides:

$$(a+b)^2 = (30-c)^2$$
$$(a+b)(a+b) = (30-c)(30-c)$$
$$a^2 + 2ab + b^2 = 900 - 60c + c^2$$

[Cancel out $a^2+b^2=c^2$ as it equals the same thing - Pythagoras' theorem.]

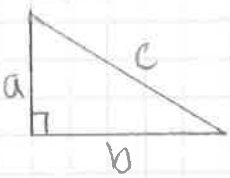
$$\Rightarrow 2ab = 900 - 60c \quad)\div 2$$
$$ab = 450 - 30c \quad)\div 2 \text{ (to find the area)}$$
$$\frac{ab}{2} = 225 - 15c \text{ units}^2$$

This can be proven in a similar way also as it's the pythagorean triple of 5, 12, 13.

$$\Rightarrow \frac{ab}{2} = 225 - 15c$$
$$\frac{(5 \times 12)}{2} = 225 - (15 \times 13)$$
$$60 \div 2 = 225 - 195$$
$$30 = 30 \text{ (units)}$$

$\therefore$ $225 - 15c = \tfrac{1}{2}ab$ (the area).

Extension - Can you find a general expression for the area when c = hypotonuse and p = perimeter? (Right angled triangle)



$$P = a+b+c \qquad A = \tfrac{1}{2}ab$$

$$-c \; \left(\begin{array}{l} a+b+c = p \\ a+b = p - c \end{array} \right.$$

(In general terms, it's used to get a+b alone)

Square both sides: $(a+b)^2 = (p-c)^2$

$$(a+b)(a+b) = (p-c)(p-c)$$
$$a^2 + 2ab + b^2 = p^2 - 2cp + c^2$$

[Cancel out $a^2+b^2=c^2$ as it's equivalent].

$$\Rightarrow \cancel{a^2} + 2ab + \cancel{b^2} = p^2 - 2cp \cancel{+c^2}$$
$$\div 2 \; \left(\begin{array}{l} 2ab = p^2 - 2cp \\ ab = \frac{p^2 - 2cp}{2} \end{array} \right. \quad)\div 2 \text{ (to get area)}$$
$$\tfrac{1}{2}ab = \frac{p^2 - 2cp}{4}$$

$\therefore$ The area of a right angled triangle when p = perimeter: $\frac{p^2 - 2cp}{4}$ (units^2)