Mathematical appendix for *The financial ether*

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1 Mathematical appendix: A more quantitative description of the economic analogy

Warning: This appendix is designed only for people with the right mathematical background.

Imagine that we have "countries" arranged on a d dimensional lattice labeled by points $\vec{n} = (n_1, n_2, \cdots, n_d)$, where each n_i is an integer number. For our spacetime we would take $d = 4$. Each point in the lattice is a country and is labeled by the vector $\vec{n}$. Let us consider the country sitting at the point $\vec{n}$ and its neighbor in the i^{th} direction, sitting at $\vec{n} + \vec{e}_i$, with $\vec{e}_i = (0, \cdots, 0, 1, 0 \cdots 0)$ where the 1 is at the i^{th} place. Here i ranges from one to d. The exchange rate between these two countries can be written as

$$R_{\vec{n},i} = e^{A_i(\vec{n})}$$

Here $R_{\vec{n},i}$ is the exchange rate and $A_j(n)$ is simply its logarithm, which we introduce for later convenience. If the country at point $\vec{n}$ uses Pesos and the country at point $\vec{n} + \vec{e}_j$ uses Dollars, then $R_{\vec{n},j}$ tells us how many Dollars you get for one Peso.

Now a gauge transformation at point $\vec{n}$ changes the local currency by multiplying it by a factor $f(n)$, which we write as

$$f(\vec{n}) = e^{\epsilon(\vec{n})}$$

This changes all the exchange rates connected to this point. More explicitly, under arbitrary currency unit transformations the exchange rates change as

$$R_{\vec{n},i} \to \frac{1}{f_{\vec{n}}} f_{\vec{n}+\vec{e}_i} R_{\vec{n},i} \ , \qquad \text{or} \qquad A_i(\vec{n}) \to A_i(\vec{n}) + \epsilon(\vec{n} + \vec{e}_i) - \epsilon(\vec{n}) \qquad (1.1)$$

When we go though an elementary basic square circuit, see figure 1, the gain factor is given by

$$\text{gain} = R_{\vec{n},i} R_{\vec{n}+\vec{e}_i,j} \frac{1}{R_{\vec{n}+\vec{e}_j,i}} \frac{1}{R_{\vec{n},j}} = e^{F_{ij}(\vec{n})}$$

$$F_{ij}(\vec{n}) = A_j(\vec{n} + \vec{e}_i) - A_j(\vec{n}) - [A_i(\vec{n} + \vec{e}_j) - A_i(\vec{n})] \qquad (1.2)$$

where we defined the "magnetic flux" F_{ij} for the corresponding elementary square circuit. Note that the gain factor, or the magnetic flux, is invariant under the change of currency, or gauge transformation, given in equation (1.1). When the gain factor is less than one you are losing money.

We now consider the same but with the addition of gold. Let us write the price of gold as

$$p(\vec{n}) = e^{\varphi(\vec{n})} \qquad (1.3)$$

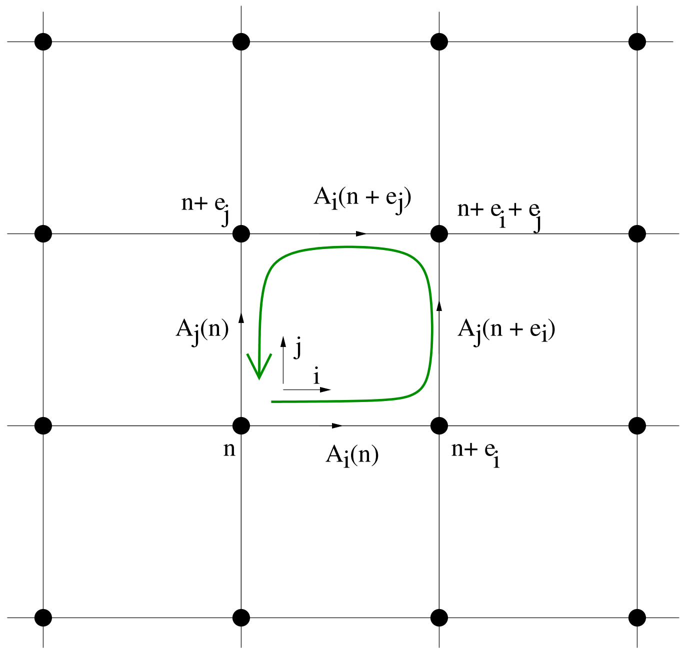

Figure 1: Elementary monetary speculative circuit. We start with some money at the country at position $\vec{n}$. First we move to its neighbor in the position $\vec{n}+\vec{e}_i$. Then we move to another neighbor at $\vec{n}+\vec{e}_i+\vec{e}_j$. Then to its neighbor at $\vec{n}+\vec{e}_j$. Finally we return to the original country. Following this circuit, and carrying only money, we can earn a profit given by the magnetic flux, (1.2).

Now there is a new opportunity to speculate by taking gold and bringing back money, see figure 2. The gain is

$$\text{gain} = \frac{p(\vec{n}+\vec{e}_i)}{p(\vec{n})R_{\vec{n},i}} = e^{D_i\varphi(n)} \ , \qquad D_i\varphi(n) \equiv \varphi(\vec{n}+\vec{e}_i) - \varphi(\vec{n}) - A_i(\vec{n}) \tag{1.4}$$

where we have defined $D_i\varphi(n)$, which in physics is called the "gauge invariant gradient of the field φ". We will call this the "gold gradient". It parametrizes the effective gain of the gold circuit, just as the magnetic flux (1.2) was parametrizing the gain of the monetary circuit.

1.1 Quantum mechanical version

We can define a probabilistic version of the above model by assuming that the exchange rates are random. They are random variables drawn from a probability distribution that depends on the magnetic fluxes. We further assume that the probability distribution has a local form, so that the probabilities of separated circuits simply multiply as if they were independent events. The price of gold is also a random variable. There is a probability for the money circuit and also for the gold circuit, see figures 1, 2.

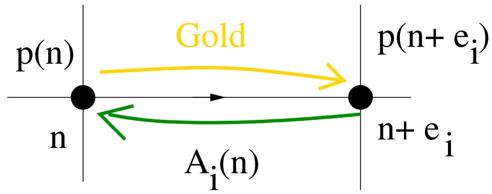

Figure 2: Elementary speculative gold circuit. We start from the country at position $\vec{n}$ and buy Gold. We take it to the neighboring country at $\vec{n}+\vec{e}_j$. We sell it there. We bring back the money to the original country. Here Gold is yellow and money is green.

More concretely, the probability for a given set of exchange rates and gold prices is

$$P[A,\varphi] = \prod_{\vec{n},i,j} \mu(F_{ij}(\vec{n})) \prod_{n,i} \tilde{\mu}(D_i\varphi(n)) \tag{1.5}$$

The product runs over all countries, which are labeled by $\vec{n}$. For each country we also multiply over all the elementary money and gold circuits that pass through that country. Here $\mu(Y)$ and $\tilde{\mu}(Y)$ are functions which are both peaked at zero. This gives the highest probability to the case where we have no opportunity to speculate. We will further assume that these probabilities can be written as

$$\mu(Y) \sim e^{-Y^2+\cdots} \ , \qquad \tilde{\mu}(Y) = e^{-\sigma^2 Y^2+\cdots}$$

where the dots represent higher order terms will not be important in the continuum limit. Here σ is just a parameter. In this situation the probability distribution (1.5) simplifies to

$$P[A,\varphi] = e^{-\sum_n \left[\sum_{i,j} F_{ij}(\vec{n})^2 + \sigma^2 \sum_i (D_i\varphi(\vec{n}))^2\right]} \tag{1.6}$$

In physics this gives the actual probability for finding the corresponding magnetic potentials in the vacuum through the following procedure. We consider a four dimensional Euclidean lattice of this form. We focus on the particular set of exchange rates and gold prices sitting at $n_4 = 0$. This is a particular three dimensional sublattice of the original lattice of countries. This surface can be interpreted as a discrete version of physical space at some instant of time, say at $t = 0$. The probabilities for the links at $n_4 = 0$ in the above Euclidean model are essentially the same as the full quantum mechanical probabilities for measuring the corresponding values of the magnetic potentials at an instant of time, say $t = 0$, in the vacuum.

We say "essentially" because we still need to take the continuum limit, indicated in figure ??. This can be done as follows. We imagine that each point in space is given by $\vec{x} = a\vec{n}$ where a is a very small number that goes to zero and $\vec{n}$ goes to infinity so that $\vec{x}$ stays fixed. In this situation it is convenient to introduce a new magnetic potential , $\mathcal{A}_i(\vec{x})$, that will stay fixed as we take the continuum limit

$$A_i(\vec{n}) \to a\mathcal{A}_i(\vec{x}) \ , \qquad \varphi(\vec{n}) \to \phi(\vec{x})$$

We also have

$$
\begin{aligned}
&F_{ij}(\vec{n}) \to a^2 \mathcal{F}_{ij}(x) \ , \qquad \mathcal{F}_{ij}(\vec{x}) \equiv \frac{\partial \mathcal{A}_j}{\partial x^i} - \frac{\partial \mathcal{A}_i}{\partial x^j} \\
&D_i \varphi(\vec{n}) \to a D_i \phi(x) = a \left[\frac{\partial \phi(\vec{x})}{\partial x^i} - \mathcal{A}_i(\vec{x}) \right] \\
&P[\mathcal{A}, \varphi] = e^{-\int dx^d \left[\sum_{i,j} \mathcal{F}_{ij}^2 + m^2 \sum_i (D_i \phi)^2 \right]} \qquad m^2 \equiv \frac{\sigma^2}{a^2}
\end{aligned}
\tag{1.7}
$$

Now, this is the full story with a massive spin one field. This is the real physical theory. As you see it is not that complicated. We have obtained it from a relatively simple probabilistic economic model. What is somewhat complicated is to relate this description to actual physical measurements. The final parameter m is the mass of the spin one particle. We recover ordinary electromagnetism by setting $m = 0$.

As we have said we can get the quantum mechanical probabilities for the vacuum from (1.7) by fixing the $\mathcal{A}_i(x)$ fields at one instant of Euclidean time $x_d = 0$ and then integrating out over the fields at all other times. The description of quantum mechanical processes involving time, such as observations at different values of ordinary time, is more complicated and would require a longer discussion of quantum mechanics. But conceptually, it is basically the same as what we have done so far. The rest of the forces of particle physics, the weak force and the strong force can be incorporated through a similar description. Some details are somewhat complicated due to the need to incorporate variables that are anticommuting "numbers" to describe electrons, neutrinos or quarks.

Finally, Maxwell's equations can be simply derived from the condition that the exchange rate variables $\mathcal{A}_i(x)$ are such that they minimize the probability. Even though the fundamental description is random we can try to find the average exchange rates and gold prices that maximizes the probability (1.6) or (1.7). These are obtained by taking the derivative of the exponent in (1.6) with respect to $A_i(\vec{n})$ and $\varphi(\vec{n})$ and setting these derivatives to zero. This gives a discrete version of the classical equations of motion for a massive field. Doing this directly in the continuum we get the standard continuum euclidean equations

$$
\sum_{j=1}^{d} \frac{\partial \mathcal{F}_{ij}}{\partial x^j} - m^2 D_i \phi = 0 \ , \qquad \frac{\partial D_i \phi}{\partial x^i} = 0 \tag{1.8}
$$

1.2 Another amusing way to obtain the classical equation in the economic model

Here we derive the classical equations of motion directly in the original economic model without going through the probabilistic interpretation. If we started with an arbitrary configuration of exchange rates and gold prices, we expect that speculators would start moving around and earning money in the process. Let us focus on one of the banks that sits between two neighboring countries. Let us say the currency of one is Pesos and the other is Dollars. If there are more speculators trying to buy Dollars than there are trying to buy Pesos, then, in the real world, the bank would try to change the exchange rate so that there is no imbalance.

In order to model this situation, we assume that the magnetic flux (1.2) or gold price gradient (1.4) are both very small. We also assume that the exchange rates are very close to one to one, and that gold prices are all very similar. In this world, the opportunities to speculate are very small. We also assume that speculators follow only the two elementary circuits in figures 1, 2. Of course, they can start from any country. We also make the important assumption the total amount of money carried by speculators following each circuit is proportional to the gain of each circuit

$$\text{money carried by speculators} = (\text{constant}) F_{ij} \qquad \text{or} \qquad = (\text{constant}) D_i \varphi$$

This statement is a bit ambiguous because we did not specify the currency. However, we have assumed that all exchange rates are close to one to one, therefore the units do not matter for small values of these exchange rates. We also restrict the gauge transformation parameters $\epsilon(\vec{n})$ to be small. When we say that the speculators carry an amount of money proportional to the flux, this money can be specified in any of the currencies on the circuit. It does not make a difference when we work to first order in the fluxes. As the speculators go around the circuit, they will make a small profit proportional to the magnetic field $F_{ij}(\vec{n})$. As a consequence of our assumptions, this is small compared to the initial amount. In other words, it is a very small percentage.

In this situation the net amount of money flowing through a given bank, say the bank that sits between the countries at point $\vec{n}$ and $\vec{n} + \vec{e}_i$, is proportional to the number of speculators crossing between these countries. Of course, speculators crossing from $\vec{n} + \vec{e}_i$ to $\vec{n}$ count with a minus sign. We want this net flux of money to be zero so that the bank does not run out of either of the two currencies. Taking into account both the monetary circuit and the gold circuit this imposes the condition

$$\sum_{j=1}^{d} (F_{i,j}(\vec{n}) - F_{i,j}(\vec{n} - \vec{e}_j)) - D_i \varphi(\vec{n}) = 0 \tag{1.9}$$

Note that all the elementary circuits that share the link going from $\vec{n}$ to $\vec{n} + \vec{e}_i$ appear in this sum.

Similarly we assume that the price of gold at each country adjusts so that there is no net gold inflow or outflow. Otherwise the inhabitants of this country would change their price of gold. This implies

$$\sum_{i=1}^{d} D_i \varphi(\vec{n}) - D_i \varphi(\vec{n} - \vec{e}_i) = 0 \tag{1.10}$$

This is summing over the contributions from speculators following the elementary Gold circuits along all the bridges connected to a given country.

These equations, (1.10) and (1.9), become (1.8) in the continuum limit. These are the magnetic part of Maxwell's equations in a time independent case. In this model, the Maxwell equations arise from the behavior of speculators that are present at short distances. In physics, there are theories where the equations of electromagnetism arise from the presence of a large number of very massive charged fields. At long distances the effect of such particles is to induce the equations of electromagnetism.

1.3 Introducing time

So far, we have been discussing the model in Euclidean space, ignoring the time direction, or taking it to be equivalent to the space dimensions. Let us now include it more properly. We can think of one of the directions in our lattice as a time direction, say it is the d^{th} dimension, and label it by the index t. The exchange rate in the time direction is simply saying that if you have some amount of money at some instant in time, then at the next instant your money will be converted to $e^{A_t(\vec{n})}$ times your original amount. Each instant in time has its own currency. Equivalently, you can think of $A_t(\vec{n})$ as the central bank interest rate of the corresponding country. And you are *required* to deposit your money in the central bank. Of course there can be opportunities to speculate by going around circuits that have one side along the time direction. You might say that it is impossible to travel backwards in time. However, you can do the following. See figure 3. You make an arrangement with another speculator. You borrow some money and give it to him. Now you have debt and he has money. He stays in the original country and you move to the neighboring country at the initial instant of time, you wait there till the next instant (depositing the money in the corresponding central bank), and then you return to the original country. If you did this properly, and if F_{ti} is non-zero, he would end up with more money that your debt. You can cancel the debt and share the profits with your friend. In physics, this would be analogous to a situation where you create an electron an positron pair at some point in spacetime with some initial velocities so that they run away from each other. Then the electric field pushes them back together at a later instant in time.

TimeCircuit.pdf

Figure 3: (a) Economic model when we include the time direction. The vertical direction is time. Here green is money and red is debt. Following this circuit there can be gain if F_{ti} is positive. (b) Corresponding process in real electromagnetism. Here some external agent creates a positron and an electron moving in opposite directions. The electric field curves their trajectories and makes them meet again. In this process there is a net "gain". When particles try to take advantage of this gain, they end up moving as if they were accelerated by the electric field.

Let us now derive Maxwell's equations. Let us consider the case with no gold, so that speculators can only carry money between different countries. We assume that we start with a spatial lattice as before. For our spacetime we would start from a three dimensional lattice of countries. Let us take time to be continuous, and think of A_t as the central bank exchange rate. For simplicity, let us choose the currency of each country so that we can set $A_t = 0$. This is like making a continuous adjustment of the currency units. As before, we assume that the amount of money that speculators carry per unit time around the elementary spatial circuits is proportional to the magnetic flux of the spatial circuit. If the net flux of money at a bank is non-zero, then the bank starts accumulating one of the two currencies. It will have an imbalance. The imbalance at the particular bank sitting between the countries at $\vec{n}$ and $\vec{n} + \vec{e}_i$ is changing as

$$\frac{dI_i(\vec{n})}{dt} = -\sum_{j=1}^{d-1} [F_{i,j}(\vec{n}) - F_{i,j}(\vec{n} - \vec{e}_j)] \tag{1.11}$$

Here $I_i(\vec{n})$ is the total imbalance of the bank. It is the difference between the amount of currency of the country at $\vec{n} + \vec{e}_i$ minus the total amount currency of the country at $\vec{n}$ that the bank has. Now we add a new rule. We assume that when the bank sees an imbalance $I_i(\vec{n})$ it starts changing the exchange rate with a speed which is proportional to the imbalance

$$\frac{dA_i(\vec{n})}{dt} = I_i(\vec{n}) \tag{1.12}$$

By taking a time derivative of (1.12) and using (1.11) we obtain

$$\frac{d^2 A_i(\vec{n})}{dt^2} = -\sum_{j=1}^{d-1} [F_{i,j}(\vec{n}) - F_{i,j}(\vec{n} - \vec{e}_j)] \tag{1.13}$$

which becomes of the Maxwell's equations in the continuum limit. This is a wave equation which describes the electromagnetic waves. The other equation, which is Gauss's law, says

$$\sum_i \frac{dA_i(\vec{n})}{dt} - \frac{dA_i(\vec{n} - \vec{e}_i)}{dt} = 0 \tag{1.14}$$

This can be derived by assuming that there are speculators going around the time circuits, see figure (3). These speculators also carry an amount of money proportional to the gain on the circuit. The gain is proportional to $\frac{dA_i}{dt}$. Demanding that there is no net amount of money deposited at each of the countries central banks imposes (1.14). We can restore a generic value of A_t by replacing $\frac{dA_i(\vec{n})}{dt} \to F_{ti} = \frac{dA_i(\vec{n})}{dt} - [A_t(\vec{n} + \vec{e}_i) - A_t(\vec{n})]$ in the above equations.

With similar assumptions we get the equation for a massive field when we include gold. With gold one needs to assume that the price of gold obeys an equation of the form $\frac{dp(\vec{n})}{dt} = -G(n)$ where $G(n)$ is the amount of gold at each country, etc.