

What's possible alternative

I know that:

$$O^2 - E^2 = O^2$$

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Let n, a, b, k be integers.

I also know that $(n + 1)^2 - n^2 = 2n + 1$, so all odd numbers can be made as a difference of the squares of consecutive numbers.

Consider the difference between two squares of even numbers:

$$\begin{aligned}(2a)^2 - (2b)^2 &= 4a^2 - 4b^2 \\ &= 4(a + b)(a - b)\end{aligned}$$

so must be a multiple of 4.

Consider the difference between two squares of odd numbers:

$$\begin{aligned}(2a + 1)^2 - (2b + 1)^2 &= 4a^2 + 4a + 1 - 4b^2 - 4b - 1 \\ &= 4(a^2 + a - b^2 - b) \\ &= 4(a - b)(a + b + 1)\end{aligned}$$

Therefore this is also a multiple of 4 (last step not necessary to see this). The fully factorised form can be used to show that this is actually a multiple of 8.

Therefore any even number which is **not** a multiple of 4 **cannot** be written as the difference of two squares.

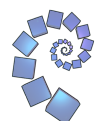
What is left to prove is that any multiple of 4 **can** be written as difference of two squares.

If the number has the form $4(2k + 1)$, then we can do it as we can write $2k + 1$ as a difference of two squares.

$$\begin{aligned}4(2k + 1) &= 4((k + 1)^2 - k^2) \\ &= (2(k + 1))^2 - (2k)^2\end{aligned}$$

Note - this is the difference between two consecutive even numbers!

If the number has the form $4 \times 2k$, i.e. it is $8k$, then we cannot have $8k = (2a)^2 - (2b)^2$, as $(a + b)(a - b)$ is either odd or a multiple of 4.



Lets try the difference between two consecutive odd numbers:

$$\begin{aligned}(2k + 1)^2 - (2k - 1)^2 &= (4k^2 + 4k + 1) - (4k^2 - 4k + 1) \\ &= 8k\end{aligned}$$

If you want to avoid negative numbers you can consider $(2k + 3)^2 - (2k + 1)^2$ and get a similar result.

Therefore:

- Every odd number can be written as the difference of the squares of two consecutive numbers
- Every even number which is an odd multiple of 4 (i.e. $4(2k + 1)$) can be written as the difference of squares of two consecutive even numbers
- Every even number which is an even multiple of 4 (i.e. $4 \times 2k = 8k$) can be written as the difference of squares of two consecutive odd numbers
- Every even number which is **not** a multiple of 4 cannot be written as the difference of squares of two integers.

If we allow ourselves to use halves, then every integer can be written as a difference of two squares, as we have $n = \frac{4n}{4} = \frac{4kn}{2^2}$, and we can write $4n$ as a difference of two square.

For example:

$$\begin{aligned}10 &= \frac{40}{4} \\ &= \frac{7^2 - 3^2}{4} \\ &= \frac{7^2}{4} - \frac{3^2}{4} \\ &= \left(\frac{7}{2}\right)^2 - \left(\frac{3}{2}\right)^2\end{aligned}$$

