What's possible alternative

I know that:

$$\begin{aligned}O^2 - E^2 &= O^2\\ E^2 - O^2 &= O^2\\ E^2 - E^2 &= E^2\\ O^2 - O^2 &= O^2\end{aligned}$$

Let n, a, b, k be integers.

I also know that $(n+1)^2 - n^2 = 2n+1$, so all odd numbers can be made as a difference of the squares of consecutive numbers.

Consider the difference between two squares of even numbers:

$$\begin{aligned}(2a)^2 - (2b)^2 &= 4a^2 - 4b^2\\ &= 4(a+b)(a-b)\end{aligned}$$

so must be a multiple of 4.

Consider the difference between two squares of odd numbers:

$$\begin{aligned}(2a+1)^2 - (2b+1)^2 &= 4a^2 + 4a + 1 - 4b^2 - 4b - 1\\ &= 4(a^2 + a - b^2 - b)\\ &= 4(a-b)(a+b+1)\end{aligned}$$

Therefore this is also a multiple of 4 (last step not necessary to see this). The fully factorised form can be used to show that this is actually a multiple of 8.

Therefore any even number which is **not** a multiple of 4 **cannot** be written as the difference of two squares.

What is left to prove is that any multiple of 4 **can** be written as difference of two squares.

If the number has the form $4(2k+1)$, then we can do it as we can write $2k+1$ as a difference of two squares.

$$\begin{aligned}4(2k+1) &= 4\big((k+1)^2 - k^2\big)\\ &= \big(2(k+1)\big)^2 - \big(2k\big)^2\end{aligned}$$

Note - this is the difference between two consecutive even numbers!

If the number has the form $4 \times 2k$, i.e. it is $8k$, then we cannot have $8k = (2a)^2 - (2b)^2$, as $(a+b)(a-b)$ is either odd or a multiple of 4.

Lets try the difference between two consecutive odd numbers:

$$\begin{aligned}(2k+1)^2-(2k-1)^2 &= (4k^2+4k+1)-(4k^2-4k+1)\\ &= 8k\end{aligned}$$

If you want to avoid negative numbers you can consider $(2k+3)^2-(2k+1)^2$ and get a similar result.

Therefore:

- Every odd number can be written as the difference of the squares of two consecutive numbers
- Every even number which is an odd multiple of 4 (i.e. $4(2k+1)$) can be written as the difference of squares of two consecutive even numbers
- Every even number which is an even multiple of 4 (i.e. $4\times 2k = 8k$) can be written as the difference of squares of two consecutive odd numbers
- Every even number which is **not** a multiple of 4 cannot be written as the difference of squares of two integers.

If we allow ourselves to use halves, then every integer can be written as a difference of two squares, as we have $n = \dfrac{4n}{4} = \dfrac{4kn}{2^2}$, and we can write $4n$ as a difference of two square.

For example:

$$\begin{aligned}10 &= \frac{40}{4}\\ &= \frac{7^2-3^2}{4}\\ &= \frac{7^2}{4}-\frac{3^2}{4}\\ &= \left(\frac{7}{2}\right)^2-\left(\frac{3}{2}\right)^2\end{aligned}$$